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Between Open Sets and Semi-Open Sets

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Abstract

We introduce and investigate ω_s-open sets as a new class of sets which lies strictly between open sets and semi-open sets. Then we use ω_s-open sets to introduce ω_s-continuous functions as a new class of functions between continuous functions and semi-continuous functions. We give several results and examples regarding our new concepts. In particular, we obtain some characterizations of ω_s-continuous functions.

Keywords: Semi-open set; ω-open set; Semi-continuous function

Introduction

Let (X,τ) be a topological space and $A \subseteq X$. We will denote the complement of A in X, the closure of A, the interior of A, the exterior of A, and the relative topology on A, by $X - A$, $\overline{A}$, Int(A), Ext(A), and τ_A, respectively. In 1963, Levine [7] defined semi-open sets as a class of sets containing the open sets as follows: A is semi-open if there exists an open set U such that $U \subseteq A \subseteq \overline{U}$, this is equivalent to say that $A \subseteq \overline{\text{Int}(A)}$. Using semi-open sets he also generalized continuity by semi-continuity as follows: A function $f:(X,\tau_1) \to (Y,\tau_2)$ is semi-continuous if for all $V \in \tau_2$, the preimage $f^{-1}(V) \in \text{SO}(X,\tau_1)$. The complement of a semi-open set is called semi-closed [5]. A point $x \in X$ is called a condensation point [6] of A if for every $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. Hdeib [6] defined ω-closed sets and ω-open sets as follows: A is called ω-closed if it contains all its condensation points. The complement of an ω-closed set is called ω-open. The collection of all ω-open sets of a topological space (X,τ) will be denoted by τ_ω. In [1], the author proved that (X,τ_ω) is a topological space and $\tau \subseteq \tau_\omega$. Moreover, it was observed that A is ω-open if and only if for every x in A there is an open set U and a countable subset C such that $x \in U - C \subseteq A$. The ω-closure of A in (X,τ), denoted by $\overline{A}^{\omega}$, is the smallest ω-closed set in (X,τ) that contains A (cf. [1]). The ω-interior of

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A in (X,τ), denoted by $\mathrm{Int}_\omega(A)$, is the largest ω-open set in (X,τ) contained in A. The ω-exterior of A in (X,τ), denoted by $\mathrm{Ext}_\omega(A)$, is defined to be $\mathrm{Int}_\omega(X-A)$. It is clear that the ω-closure (resp. ω-interior) of A in (X,τ) equals the closure (resp. interior) of A in (X,τ_ω). In 2002, Al-Zoubi and Al-Nashef [2] used ω-open sets to define semi ω-open sets as a weaker form of semi-open sets as follows: A is semi ω-open if there exists an ω-open set U such that $U\subseteq A\subseteq \overline{U}$. The collection of all semi ω-open sets of a topological space (X,τ) will be denoted by $\mathrm{S}\omega\mathrm{O}(X,\tau)$. Al-Zoubi [4] used semi ω-open sets to introduce semi ω-continuous functions as a weaker form of ω-continuous functions as follows: A function $f:(X,\tau_1)\to(Y,\tau_2)$ is semi ω-continuous [4] if for all $V\in\tau_2$, the preimage $f^{-1}(V)\in \mathrm{S}\omega\mathrm{O}(X,\tau_1)$. This paper is devoted to define ω_s-opennes as a property of sets that is strictly weaker than openness and stronger than semi-openness as follows: A is ω_s-open if there exists an open set U such that $U\subseteq A\subseteq \overline{U}^{\omega}$. We investigate this class of sets, and use it to study a new property of functions strictly between continuity and semi-continuity, and another new property of functions strictly between slight continuity and slight semi-continuity.

Throughout this paper $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{Q}^c$, will denote the set of real numbers, the set of natural numbers, the set of rational numbers, and the set of irrational numbers, respectively. For any non-empty set X we denote by τ_{disc} the discrete topology on X. Finally, by τ_{u} we mean the usual topology on $\mathbb{R}$.

The following sequence of theorems will be useful in the sequel:

Theorem 1.1 ([3]). *Let (X,τ) be a topological space and $A\subseteq X$. Then*

(a) *If A is non-empty, then $(\tau_A)_\omega=(\tau_\omega)_A$.*

(b) $(\tau_\omega)_\omega=\tau_\omega$.

Theorem 1.2 ([2]). *Let (X,τ) be a topological space. Then*

(a) $\mathrm{SO}(X,\tau)\subseteq \mathrm{S}\omega\mathrm{O}(X,\tau)$, *and* $\mathrm{SO}(X,\tau)\neq \mathrm{S}\omega\mathrm{O}(X,\tau)$ *in general.*

(b) $\tau_\omega\subseteq \mathrm{S}\omega\mathrm{O}(X,\tau)$, *and* $\tau_\omega\neq \mathrm{S}\omega\mathrm{O}(X,\tau)$ *in general.*

Theorem 1.3 ([1]). *Let (X,τ) be a topological space. Then*

(a) *If (X,τ) is anti-locally countable, then $\overline{A}^{\omega}=\overline{A}$ for all $A\in\tau_\omega$, and* $\mathrm{Int}_\omega(A)=\mathrm{Int}(A)$ *for all ω-closed set A in (X,τ).*

(b) *If (X,τ) is locally countable, then τ_ω is the discrete topology.*

ω_s-Open sets

Definition 2.1. Let A of be a subset of a topological space (X,τ). Then A is called ω_s-open of (X,τ), if there exists $U\in\tau$ such that $U\subseteq A\subseteq\overline{U}^{\omega}$ and A

is called ω_s-closed if $X - A$ is ω_s-open. The family of all ω_s-open subsets of (X,τ) will be denoted by $\omega_s(X,\tau)$.

Theorem 2.2. *Let (X,τ) be a topological space. Then $\tau \subseteq \omega_s(X,\tau) \subseteq \mathrm{SO}(X,\tau)$.*

Proof. Let $A \in \tau$. Take $U = A$. Then $U \in \tau$ and $U \subseteq A \subseteq \overline{U}^{\omega}$. This shows that $A \in \omega_s(X,\tau)$. It follows that $\tau \subseteq \omega_s(X,\tau)$. Let $A \in \omega_s(X,\tau)$. Then there exists $U \in \tau$ such that $U \subseteq A \subseteq \overline{U}^{\omega}$, but $\overline{U}^{\omega} \subseteq \overline{U}$. Thus $A \in \mathrm{SO}(X,\tau)$. This shows that $\omega_s(X,\tau) \subseteq \mathrm{SO}(X,\tau)$. □

In the following example we will see that, in general, neither of the two inclusions in Theorem 2.2 are equalities:

Example 2.3. Consider $(\mathbb{R},\tau)$, where $\tau = \{\varnothing, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. It is not difficult to check that $\overline{\mathbb{N}}^{\omega} = \mathbb{N}$, $\overline{\mathbb{N}} = \mathbb{Q}$, and $\overline{\mathbb{Q}^c}^{\omega} = \mathbb{R} - \mathbb{N}$. Thus $\mathbb{Q} \in \mathrm{SO}(X,\tau) - \omega_s(X,\tau)$ and $\mathbb{R} - \mathbb{N} \in \omega_s(X,\tau) - \tau$.

Theorem 2.4. *Let (X,τ) be a topological space. Then*

(*a*) *If (X,τ) is anti-locally countable, then $\omega_s(X,\tau) = \mathrm{SO}(X,\tau)$.*
(*b*) *If (X,τ) is locally countable, then $\tau = \omega_s(X,\tau)$.*

Proof. (a) By Theorem 2.2 it is sufficient to show that $\mathrm{SO}(X,\tau) \subseteq \omega_s(X,\tau)$. Let $A \in \mathrm{SO}(X,\tau)$. Then there exists $U \in \tau$ such that $U \subseteq A \subseteq \overline{U}$. Since (X,τ) is anti-locally countable, then by Theorem 1.3 (a), $\overline{U} = \overline{U}^{\omega}$. It follows that $A \in \omega_s(X,\tau)$.

(b) By Theorem 2.2 it is sufficient to show that $\omega_s(X,\tau) \subseteq \tau$. Let us take $A \in \omega_s(X,\tau)$. Then there exists $U \in \tau$ such that $U \subseteq A \subseteq \overline{U}^{\omega}$. Since (X,τ) is locally countable, then by Theorem 1.3 (b), $\overline{U}^{\omega} = U$. It follows that $A = U$ and hence $A \in \tau$. □

The following example shows that ω-open sets and ω_s-open sets are independent:

Example 2.5. Consider $(\mathbb{R},\tau)$ where $\tau = \{\varnothing, \mathbb{R}, [0,\infty)\}$. It is not difficult to check that $\overline{[0,\infty)}^{\omega} = \mathbb{R}$. Thus $[-1,\infty) \in \omega_s(X,\tau) - \tau_\omega$ and $(0,\infty) \in \tau_\omega - \omega_s(X,\tau)$.

Theorem 2.6. *A subset A of a topological space (X,τ) is ω_s-open if and only if $A \subseteq \overline{\mathrm{Int}(A)}^{\omega}$.*

Proof. Necessity. Let A be ω_s-open. Then there exists some $U \in \tau$ such that $U \subseteq A \subseteq \overline{U}^{\omega}$. Since $U \subseteq A$, then $U = \mathrm{Int}(U) \subseteq \mathrm{Int}(A)$ and so $\overline{U}^{\omega} \subseteq \overline{\mathrm{Int}(A)}^{\omega}$. Therefore, $A \subseteq \overline{\mathrm{Int}(A)}^{\omega}$.

Sufficiency. Suppose that $A \subseteq \overline{\text{Int}(A)}^{\omega}$. Take $U = \text{Int}(A)$. Then $U \in \tau$ with $U \subseteq A \subseteq \overline{U}^{\omega}$. It follows that A is ω_s-open. □

Theorem 2.7. *Arbitrary unions of ω_s-open sets in a topological space is ω_s-open.*

Proof. Let (X, τ) be a topological space and let $\{A_\alpha : \alpha \in \Delta\} \subseteq \omega_s(X, \tau)$. For each $\alpha \in \Delta$, there exists $U_\alpha \in \tau$ such that $U_\alpha \subseteq A_\alpha \subseteq \overline{U_\alpha}^{\omega}$. So, we have $\bigcup_{\alpha \in \Delta} U_\alpha \in \tau$, with

$$\bigcup_{\alpha \in \Delta} U_\alpha \subseteq \bigcup_{\alpha \in \Delta} A_\alpha \subseteq \bigcup_{\alpha \in \Delta} \overline{U_\alpha}^{\omega} \subseteq \overline{\bigcup_{\alpha \in \Delta} U_\alpha}^{\omega}.$$

It follows that $\bigcup_{\alpha \in \Delta} A_\alpha \in \omega_s(X, \tau)$. □

Corollary 2.8. *If $\{C_\alpha : \alpha \in \Delta\}$ is a collection of ω_s-closed subsets of a topological space (X, τ), then $\bigcap \{C_\alpha : \alpha \in \Delta\}$ is ω_s-closed.*

The following example shows that the intersection of two ω_s-open sets need not to be ω_s-open in general:

Example 2.9. Consider $(\mathbb{R}, \tau_{\text{u}})$. Let $A = [0, 1]$, $B = [1, 2]$. By Theorem 1.3 (a), $\overline{(0,1)}^{\omega} = \overline{(0,1)} = A$, and $\overline{(1,2)}^{\omega} = \overline{(1,2)} = B$. Thus $A, B \in \omega_s(X, \tau)$, but $A \cap B = \{1\} \notin \omega_s(X, \tau)$.

Theorem 2.10. *For any topological space, the intersection of two ω_s-open sets where one of them is open is also ω_s-open.*

Proof. Let (X, τ) be a topological space, $A \in \tau$ and $B \in \omega_s(X, \tau)$. Choose a set $U \in \tau$ such that $U \subseteq B \subseteq \overline{U}^{\omega}$. Now we have $A \cap U \in \tau$, and then $A \cap U \subseteq A \cap B \subseteq A \cap \overline{U}^{\omega} \subseteq \overline{A \cap U}^{\omega}$. This shows that, $A \cap B \in \omega_s(X, \tau)$. □

Corollary 2.11. *For any topological space, the union of two ω_s-closed sets where one of them is closed is also ω_s-closed.*

Theorem 2.12. *Let (X, τ) be a topological space, B a non-empty subset of X and $A \subseteq B$. Then*

(*a*) *If $A \in \omega_s(X, \tau)$, then $A \in \omega_s(B, \tau_B)$.*

(*b*) *If $B \in \tau$ and $A \in \omega_s(B, \tau_B)$, then $A \in \omega_s(X, \tau)$.*

Proof. (a) Let $A \in \omega_s(X, \tau)$. Then there is $U \in \tau$ such that $U \subseteq A \subseteq \overline{U}^{\omega}$. Then $U = U \cap B \subseteq A \subseteq \overline{U}^{\omega} \cap B$. Note that $\overline{U}^{\omega} \cap B$ is the closure of U in $(\tau_\omega)_B$ and by Theorem 1.1 (a), it is the closure of U in $(\tau_B)_\omega$. This shows that $A \in \omega_s(B, \tau_B)$.

(b) Let $B \in \tau$ and $A \in \omega_s(B, \tau_B)$. Since $A \in \omega_s(B, \tau_B)$, there is $V \in \tau_B$ such that $V \subseteq A \subseteq H$ where H is the closure of V in $(B, (\tau_B)_\omega)$. Since $B \in \tau$, then $V \in \tau$. Also, $V \subseteq A \subseteq H \subseteq \overline{V}^{\omega}$. Therefore, $A \in \omega_s(X, \tau)$. □

Theorem 2.13. *Let (X,τ) be a topological space. Let $A\in\omega_s(X,\tau)$ and suppose that $A\subseteq B\subseteq\overline{A}^{\omega}$, then $B\in\omega_s(X,\tau)$.*

Proof. Since $A\in\omega_s(X,\tau)$, there exists $U\in\tau$ such that $U\subseteq A\subseteq\overline{U}^{\omega}$. Since $A\subseteq\overline{U}^{\omega}$, then $\overline{A}^{\omega}\subseteq\overline{U}^{\omega}$. Since $B\subseteq\overline{A}^{\omega}$, then $B\subseteq\overline{U}^{\omega}$. Therefore, we have $U\in\tau$ and $U\subseteq A\subseteq B\subseteq\overline{U}^{\omega}$. This shows that $B\in\omega_s(X,\tau)$. □

Theorem 2.14. *For any topological space (X,τ) we have that $\mathrm{SO}(X,\tau_\omega)=\omega_s(X,\tau_\omega)$.*

Proof. By Theorem 2.2, we have $\omega_s(X,\tau_\omega)\subseteq\mathrm{SO}(X,\tau_\omega)$. Conversely, let $A\in\mathrm{SO}(X,\tau_\omega)$, then there exists $U\in\tau_\omega$ such that $U\subseteq A\subseteq H$, where H is the closure of U in (X,τ_ω). By Theorem 1.1 (b), we have $(\tau_\omega)_\omega=\tau_\omega$ and so $H=\overline{U}^{\omega}$. It follows that $A\in\omega_s(X,\tau_\omega)$. □

Theorem 2.15. *For any topological space (X,τ) we have the relation $\tau=\{\mathrm{Int}(A):\ A\in\omega_s(X,\tau)\}$.*

Proof. It follows because from Theorem 2.2 we have $\tau\subseteq\omega_s(X,\tau)$. □

Theorem 2.16. *A subset C of a topological space (X,τ) is ω_s-closed if and only if $\mathrm{Int}_\omega(\overline{C})\subseteq C$.*

Proof. Necessity. Suppose that C is ω_s-closed in (X,τ). Then $X-C$ is ω_s-closed and by Theorem 2.6, $X-C\subseteq\overline{\mathrm{Int}(X-C)}^{\omega}$. So

$$\begin{aligned}\mathrm{Int}_\omega(\overline{C})&\subseteq\mathrm{Ext}_\omega(X-\overline{C})\\&=\mathrm{Ext}_\omega(\mathrm{Ext}(C))\\&=X-\overline{\mathrm{Ext}(C)}^{\omega}\\&=X-\overline{\mathrm{Int}(X-C)}^{\omega}\\&\subseteq C.\end{aligned}$$

Sufficiency. Suppose that $\mathrm{Int}_\omega(\overline{C})\subseteq C$. Then

$$\begin{aligned}X-C&\subseteq X-\mathrm{Int}_\omega(\overline{C})\\&=X-\mathrm{Ext}_\omega(X-\overline{C})\\&=X-\mathrm{Ext}_\omega(\mathrm{Ext}(C))\\&=\overline{(\mathrm{Ext}(C))}^{\omega}\\&=\overline{\mathrm{Int}(X-C)}^{\omega}.\end{aligned}$$

By Theorem 2.6 it follows that $X-C$ is ω_s-open, and hence C is ω_s-closed. □

Definition 2.17. Let (X,τ) be a topological space and let $A \subseteq X$.

(a) The ω_s-closure of A in (X,τ) is denoted by $\overline{A}^{\omega_s}$ and defined as follows:

$$\overline{A}^{\omega_s} = \bigcap \{C : C \text{ is } \omega_s\text{-closed in } (X,\tau) \text{ and } A \subseteq C\}.$$

(b) The ω_s-interior of A in (X,τ) is denoted by $\operatorname{Int}_{\omega_s}(A)$ and defined as follows:

$$\operatorname{Int}_{\omega_s}(A) = \bigcup \{U : U \text{ is } \omega_s\text{-open in } (X,\tau) \text{ and } U \subseteq A\}.$$

Remark 2.18. Let (X,τ) be a topological space and let $A \subseteq X$. Then

(a) $\overline{A}^{\omega_s}$ is the smallest ω_s-closed set in (X,τ) containing A.

(b) A is ω_s-closed in (X,τ) if and only if $A = \overline{A}^{\omega_s}$.

(c) $\operatorname{Int}_{\omega_s}(A)$ is the largest ω_s-open set in (X,τ) contained in A.

(d) A is ω_s-open in (X,τ) if and only if $A = \operatorname{Int}_{\omega_s}(A)$.

(e) $x \in \overline{A}^{\omega_s}$ if and only if for every $B \in \omega_s(X,\tau)$ with $x \in B$, $A \cap B \neq \varnothing$.

(f) $\operatorname{Int}_{\omega_s}(X - A) \cap \overline{A}^{\omega_s} = \varnothing$.

(g) $X = \operatorname{Int}_{\omega_s}(X - A) \cup \overline{A}^{\omega_s}$.

(h) $X - \overline{A}^{\omega_s} = \operatorname{Int}_{\omega_s}(X - A)$ and $X - \operatorname{Int}_{\omega_s}(X - A) = \overline{A}^{\omega_s}$.

Theorem 2.19. *Let $f : (X,\tau) \to (Y,\sigma)$ be an open function such that $f : (X,\tau_\omega) \to (Y,\sigma_\omega)$ is continuous. Then for every $A \in \omega_s(X,\tau)$ we have $f(A) \in \omega_s(Y,\sigma)$.*

Proof. Let $A \in \omega_s(X,\sigma)$. Then there exists $U \in \tau$ such that $U \subseteq A \subseteq \overline{U}^{\omega}$, and so $f(U) \subseteq f(A) \subseteq f(\overline{U}^{\omega})$. Since $f : (X,\tau) \to (Y,\sigma)$ is open, then $f(U) \in \sigma$. Since $f : (X,\tau_\omega) \to (Y,\sigma_\omega)$ is continuous, then $f(\overline{U}^{\omega}) \subseteq \overline{f(U)}^{\omega}$. It follows that $f(A) \in \omega_s(Y,\sigma)$. □

The condition *"open function"* cannot be dropped from Theorem 2.19 as shown by:

Example 2.20. Consider $f : (\mathbb{R},\tau_{\text{disc}}) \to (\mathbb{R},\tau_{\text{u}})$, where $f(x) = 0$ for all $x \in \mathbb{R}$. Then it is obvious that $f : (\mathbb{R},(\tau_{\text{disc}})_\omega) \to (\mathbb{R},(\tau_{\text{u}})_\omega)$ is continuous. On the other hand, $\{0\} \in \omega_s(\mathbb{R},\tau_{\text{disc}})$ but $f(\{0\}) = \{0\} \notin \omega_s(\mathbb{R},\tau_{\text{u}})$.

ω_s-Continuous functions

Definition 3.1. A function $f:(X,\tau)\to(Y,\sigma)$ is called ω_s-continuous, if for each $V\in\sigma$, the preimage $f^{-1}(V)\in\omega_s(X,\sigma)$.

Theorem 3.2. *The notions of continuity satisfy that*

(a) *Every continuous function is ω_s-continuous.*

(b) *Every ω_s-continuous function is semi-continuous.*

Proof. Theorem 2.2. □

The following example will show that the converse of each of the two implications in Theorem 3.2 is not true in general:

Example 3.3. Let $f,g:(\mathbb{R},\tau)\to(\{a,b\},\tau_{\text{disc}})$, with τ as in Example 2.3 and

$$f(x)=\begin{cases} a & \text{if } x\in\mathbb{N} \\ b & \text{if } x\in\mathbb{R}-\mathbb{N}\end{cases} \quad \text{and} \quad g(x)=\begin{cases} a & \text{if } x\in\mathbb{Q}^c \\ b & \text{if } x\in\mathbb{Q}\end{cases}.$$

Since $f^{-1}(\{a\})=\mathbb{N}\in\tau\subseteq\omega_s(\mathbb{R},\tau)$ and $f^{-1}(\{b\})=\mathbb{R}-\mathbb{N}\in\omega_s(\mathbb{R},\tau)-\tau$, then f is ω_s-continuous but not continuous. Also, Since $g^{-1}(\{a\})=\mathbb{Q}^c\in\tau\subseteq \mathrm{SO}(X,\tau)$ and $g^{-1}(\{b\})=\mathbb{Q}\in \mathrm{SO}(\mathbb{R},\tau)-\omega_s(\mathbb{R},\tau)$, then f is semi-continuous but not ω_s-continuous.

Theorem 3.4. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function.*

(a) *If (X,τ) is locally countable, then f is continuous if and only if f is ω_s-continuous.*

(b) *If (X,τ) is anti-locally countable, then f is ω_s-continuous if and only if f is semi-continuous.*

Proof. (a) It is a consequence of Theorems 2.4 (a) and 3.2 (a).

(b) It is a consequence of Theorems 2.4 (b) and 3.2 (b). □

Theorem 3.5. *A function $f:(X,\tau)\to(Y,\sigma)$ is ω_s-continuous if and only if for every $x\in X$ and every open set V containing $f(x)$ there exists $U\in\omega_s(X,\tau)$ such that $x\in U$ and $f(U)\subseteq V$.*

Proof. Necessity. Assume that $f:(X,\tau)\to(Y,\sigma)$ is ω_s-continuous. Let us take $V\in\sigma$ with $f(x)\in V$. By ω_s-continuity, $f^{-1}(V)\in\omega_s(X,\tau)$. Set $U=f^{-1}(V)$. Then $U\in\omega_s(X,\tau)$ satisfies $x\in U$ and $f(U)\subseteq V$.

Sufficiency. Let $V\in\sigma$. For each $x\in f^{-1}(V)$ we have $f(x)\in V$, and thus there exists $U_x\in\omega_s(X,\tau)$ such that $x\in U_x$, $f(U_x)\subseteq V$, and $x\in U_x\subseteq f^{-1}(V)$. Thus $f^{-1}(V)=\bigcup\{U_x : x\in f^{-1}(V)\}$. Therefore, by Theorem 2.7, it follows $f^{-1}(V)\in\omega_s(X,\tau)$. This shows that f is ω_s-continuous. □

Theorem 3.6. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following conditions are equivalent:*

(a) *The function f is ω_s-continuous.*

(b) *Inverse images of all members of a base $\mathscr{B}$ for σ are in $\omega_s(X,\tau)$.*

(c) *Inverse images of all closed subsets of (Y,σ) are ω_s-closed in (X,τ).*

(d) *For every $A\subseteq X$ we have $f(\overline{A}^{\omega_s})\subseteq\overline{f(A)}$.*

(e) *For every $B\subseteq Y$ we have $\overline{f^{-1}(B)}^{\omega_s}\subseteq f^{-1}(\overline{B})$.*

(f) *For every $B\subseteq Y$ we have $f^{-1}(\mathrm{Int}(B))\subseteq\mathrm{Int}_{\omega_s}(f^{-1}(B))$.*

Proof. (a) $\Longrightarrow$ (b). Obvious.

(b) $\Longrightarrow$ (c). Suppose $\mathscr{B}$ is a base for σ such that $f^{-1}(B)\in\omega_s(X,\tau)$ for every $B\in\mathscr{B}$. Let C be a non-empty closed subset of (Y,σ). Then $Y-C\in\tau-\{\varnothing\}$. Choose $\mathscr{B}^*\subseteq\mathscr{B}$ such that $Y-C=\bigcup\{B:B\in\mathscr{B}^*\}$. Then

$$\begin{aligned}X-f^{-1}(C)&=f^{-1}(Y-C)\\&=f^{-1}\big(\bigcup\{B:B\in\mathscr{B}^*\}\big)\\&=\bigcup\{f^{-1}(B):B\in\mathscr{B}^*\}.\end{aligned}$$

By assumption $f^{-1}(B)\in\omega_s(X,\tau)$ for every $B\in\mathscr{B}^*$, then by Theorem 2.7 we have $X-f^{-1}(C)\in\omega_s(X,\tau)$, and hence $f^{-1}(C)$ is ω_s-closed in (X,τ).

(c) $\Longrightarrow$ (d). Let $A\subseteq X$. Then $\overline{f(A)}$ is closed in (Y,σ), and by (c) $f^{-1}(\overline{f(A)})$ is ω_s-closed in (X,τ). Since $A\subseteq f^{-1}(f(A))\subseteq f^{-1}(\overline{f(A)})$, and $f^{-1}(\overline{f(A)})$ is ω_s-closed in (X,τ), then $\overline{A}^{\omega_s}\subseteq f^{-1}(\overline{f(A)})$, and thus $f(\overline{A}^{\omega_s})\subseteq f(f^{-1}(\overline{f(A)}))\subseteq\overline{f(A)}$.

(d) $\Longrightarrow$ (e). Let $B\subseteq Y$. Then $f^{-1}(B)\subseteq X$, and by (d) $f(\overline{f^{-1}(B)}^{\omega_s})\subseteq\overline{f(f^{-1}(B))}\subseteq\overline{B}$. Therefore, $\overline{f^{-1}(B)}^{\omega_s}\subseteq f^{-1}(\overline{B})$.

(e) $\Longrightarrow$ (f). Let $B\subseteq Y$. Then by (e), $\overline{f^{-1}(Y-B)}^{\omega_s}\subseteq f^{-1}(\overline{Y-B})$. Also by Theorem 2.19 (h), $X-\overline{X-f^{-1}(B)}^{\omega_s}=\mathrm{Int}_{\omega_s}(f^{-1}(B))$. Thus,

$$\begin{aligned}f^{-1}(\mathrm{Int}(B))&=f^{-1}(Y-\overline{Y-B})\\&=X-f^{-1}(\overline{Y-B})\\&\subseteq X-\overline{f^{-1}(Y-B)}^{\omega_s}\\&=X-\overline{X-f^{-1}(B)}^{\omega_s}\\&=\mathrm{Int}_{\omega_s}(f^{-1}(B)).\end{aligned}$$

□

Lemma 3.7. *Let (X,τ) be a topological space and let $A\subseteq X$. Then*

$$\overline{A}^{\omega_s}=A\cup\mathrm{Int}_{\omega}(\overline{A}).$$

Proof. Since $\overline{A}^{\omega_s}$ is ω_s-closed, then by Theorem 2.15 $\mathrm{Int}_\omega(\overline{(\overline{A}^{\omega_s})}) = \mathrm{Int}_\omega(\overline{A}^{\omega_s}) \subseteq \overline{A}^{\omega_s}$. Therefore, $\mathrm{Int}_\omega(\overline{A}) \subseteq \mathrm{Int}_\omega(\overline{(\overline{A}^{\omega_s})}) \subseteq \overline{A}^{\omega_s}$, and hence $A \cup \mathrm{Int}_\omega(\overline{A}) \subseteq \overline{A}^{\omega_s}$. To see that $\overline{A}^{\omega_s} = A \cup \mathrm{Int}_\omega(\overline{A})$, it is sufficient to show that $A \cup \mathrm{Int}_\omega(\overline{A})$ is ω_s-closed. Since $\mathrm{Int}_\omega(\overline{A}) \subseteq \overline{A}$, then $\overline{\mathrm{Int}_\omega(\overline{A})} \subseteq \overline{A}$. Therefore,

$$\begin{aligned}\mathrm{Int}_\omega(\overline{A \cup \mathrm{Int}_\omega(\overline{A})}) &= \mathrm{Int}_\omega(\overline{A} \cup \overline{\mathrm{Int}_\omega(\overline{A})})\\ &= \mathrm{Int}_\omega(\overline{A})\\ &\subseteq A \cup \mathrm{Int}_\omega(\overline{A}),\end{aligned}$$

and by Theorem 2.15 it follows that $A \cup \mathrm{Int}_\omega(\overline{A})$ is ω_s-closed. □

Theorem 3.8. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent:*

(a) *f is ω_s-continuous.*

(b) *For every $A \subseteq X$ we have $f(\mathrm{Int}_\omega(\overline{A})) \subseteq \overline{f(A)}$.*

(c) *For every $B \subseteq Y$ we have $\mathrm{Int}_\omega(\overline{f^{-1}(B)}) \subseteq f^{-1}(\overline{B})$.*

Proof. (a) $\Longrightarrow$ (b). Suppose that f is ω_s-continuous. Let $A \subseteq X$. Then by Theorem 3.6 (d), $f(\overline{A}^{\omega_s}) \subseteq \overline{f(A)}$. Therefore, by Lemma 3.7 we have

$$f(\mathrm{Int}_\omega(\overline{A})) \subseteq f(\overline{A}^{\omega_s}) \subseteq \overline{f(A)}.$$

(b) $\Longrightarrow$ (a). We will apply Theorem 3.6 (d). Let $A \subseteq X$. Then by (b), we have $f(\mathrm{Int}_\omega(\overline{A})) \subseteq \overline{f(A)}$. Also, we have $f(A) \subseteq \overline{f(A)}$ always. Therefore, by Lemma 3.7 we have

$$\begin{aligned}f(\overline{A}^{\omega_s}) &= f(A \cup \mathrm{Int}_\omega(\overline{A}))\\ &= f(A) \cup f(\mathrm{Int}_\omega(\overline{A}))\\ &\subseteq \overline{f(A)}.\end{aligned}$$

(a) $\Longrightarrow$ (c). Suppose that f is ω_s-continuous. Let $B \subseteq Y$. Then by Theorem 3.6 (e), $\overline{f^{-1}(B)}^{\omega_s} \subseteq f^{-1}(\overline{B})$. Therefore, by Lemma 3.7 we have

$$\mathrm{Int}_\omega(\overline{f^{-1}(B)}) \subseteq \overline{f^{-1}(B)}^{\omega_s} \subseteq f^{-1}(\overline{B}).$$

(c) $\Longrightarrow$ (a). We will apply Theorem 3.6 (e). Let $B \subseteq Y$. Then by (c), we have $\mathrm{Int}_\omega(\overline{f^{-1}(B)}) \subseteq f^{-1}(\overline{B})$. Also, we have $f^{-1}(B) \subseteq f^{-1}(\overline{B})$ always. Therefore, by Lemma 3.7 we have

$$\overline{f^{-1}(B)}^{\omega_s} = f^{-1}(B) \cup \mathrm{Int}_\omega(\overline{f^{-1}(B)}) \subseteq f^{-1}(\overline{B}).$$ □

Theorem 3.9. *If $f:(X,\tau)\to(Y,\sigma)$ is ω_s-continuous and $g:(Y,\sigma)\to(Z,\lambda)$ is continuous, then $g\circ f:(X,\tau)\to(Z,\lambda)$ is a ω_s-continuous.*

Proof. Let $V\in\lambda$. Since g is continuous, then $g^{-1}(V)\in\sigma$. Since f is ω_s-continuous, then $(g\circ f)^{-1}(V)=f^{-1}(g^{-1}(V))\in\omega_s(X,\tau)$. □

In general, the composition of two ω_s-continuous functions does not need to be ω_s-continuous as the following example clarifies:

Example 3.10. Let $f,g:(\mathbb{R},\tau_u)\to(\mathbb{R},\tau_u)$, where

$$f(x)=\begin{cases} x & \text{if } x\leq 1\\ 0 & \text{if } x>1\end{cases},\quad \text{and}\quad g(x)=\begin{cases} 0 & \text{if } x<1\\ 3 & \text{if } x\geq 1\end{cases}.$$

Then

$$(g\circ f)(x)=\begin{cases} 0 & \text{if } x\neq 1\\ 3 & \text{if } x=1\end{cases}.$$

Since f and g are obviously semi-continuous and $(\mathbb{R},\tau_u)$ is anti-locally countable, then by Theorem 3.4 (b) f and g are ω_s-continuous. On the other hand, since $(2,\infty)\in\tau_u$ but $(g\circ f)^{-1}(2,\infty)=\{1\}\notin\omega_s(\mathbb{R},\tau_u)$, then $g\circ f$ is not ω_s-continuous.

Theorem 3.11. *Let $\{f_\alpha:(X,\tau)\to(Y_\alpha,\sigma_\alpha)\}_{\alpha\in\Delta}$ be a family of functions. If the function $f:(X,\tau)\to(\prod_{\alpha\in\Delta}Y_\alpha,\tau_{\text{prod}})$ defined by $f(x)=(f_\alpha(x))_{\alpha\in\Delta}$ is ω_s-continuous, then f_α is ω_s-continuous, for every $\alpha\in\Delta$.*

Proof. Suppose that f is ω_s-continuous and let $\beta\in\Delta$. Then $f_\beta=\pi_\beta\circ f$ where $\pi_\beta:(\prod_{\alpha\in\Delta}Y_\alpha,\tau_{\text{prod}})\to(Y_\beta,\sigma_\beta)$ is the projection function on Y_β. Since π_β is continuous, then by Theorem 3.9, f_β is ω_s-continuous. □

The following example will show that the converse of Theorem 3.11 is not true in general:

Example 3.12. Define $f,g:(\mathbb{R},\tau_u)\to(\mathbb{R},\tau_u)$, and $h:(\mathbb{R},\tau_u)\to(\mathbb{R}\times\mathbb{R},\tau_{\text{prod}})$ by

$$f(x)=\begin{cases} 2 & \text{if } x\leq 0\\ -2 & \text{if } x>0\end{cases},\quad g(x)=\begin{cases} -2 & \text{if } x<0\\ 2 & \text{if } x\geq 0\end{cases},\quad \text{and}\quad h(x)=(f(x),g(x)).$$

Since f and g are obviously semi-continuous, and $(\mathbb{R},\tau_u)$ is anti-locally countable, then by Theorem 3.4 (b) f and g are ω_s-continuous. On the other hand, since $(0,\infty)\times(-\infty,0)\in\tau_{\text{prod}}$ but $h^{-1}\big((0,\infty)\times(-\infty,0)\big)=\{0\}\notin\omega_s(\mathbb{R},\tau_u)$, then h is not ω_s-continuous.

Theorem 3.13. *Let* $\{f_\alpha : (X,\tau) \to (Y_\alpha, \sigma_\alpha)\}_{\alpha\in\Delta}$ *be a family of functions. If* f_{α_0} *is* ω_s*-continuous for some* $\alpha_0 \in \Delta$*, and if* f_α *is continuous for all* $\alpha \in \Delta - \{\alpha_0\}$*, then the function* $f : (X,\tau) \to (\prod_{\alpha\in\Delta} Y_\alpha, \tau_{\text{prod}})$ *defined by* $f(x) = (f_\alpha(x))_{\alpha\in\Delta}$ *is* ω_s*-continuous.*

Proof. We will apply statement (b) of Theorem 3.6. Let A be a basic open set of $(\prod_{\alpha\in\Delta} Y_\alpha, \tau_{\text{prod}})$, without loss of generality we may assume that $A = \pi_{\alpha_0}^{-1}(U_{\alpha_0}) \cap \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$, where U_{α_i} is a basic open set of Y_{α_i} for all $i = 0, 1, \ldots, n$. Then

$$\begin{aligned} f^{-1}(A) &= ((\pi_{\alpha_0} \circ f)^{-1}(U_{\alpha_0})) \cap ((\pi_{\alpha_1} \circ f)^{-1}(U_{\alpha_1})) \cap \cdots \cap ((\pi_{\alpha_n} \circ f)^{-1}(U_{\alpha_n})) \\ &= (f_{\alpha_0}^{-1}(U_{\alpha_0})) \cap \left[(f_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \cdots \cap (f_{\alpha_n}^{-1}(U_{\alpha_n}))\right]. \end{aligned}$$

By assumption $f_{\alpha_0}^{-1}(U_{\alpha_0}) \in \omega_s(X,\tau)$ and $f_{\alpha_i}^{-1}(U_{\alpha_i}) \in \tau$ for all $i = 0, 1, \ldots, n$. Thus $(f_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \cdots \cap (f_{\alpha_n}^{-1}(U_{\alpha_n})) \in \tau$, and by Theorem 2.9, we have that $f^{-1}(A) \in \omega_s(X,\tau)$. It follows that f is ω_s-continuous. □

Corollary 3.14. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be a function and denote by* $g : (X,\tau) \to (X \times Y, \tau_{\text{prod}})$ *the graph function of* f *given by* $g(x) = (x, f(x))$*, for every* $x \in X$*. Then* g *is* ω_s*-continuous if and only if* f *is* ω_s*-continuous.*

Proof. Necessity. Suppose that g is ω_s-continuous. Then by Theorem 3.11, f is ω_s-continuous.

Sufficiency. Suppose that f is ω_s-continuous. Note that $h(x) = (I(x), f(x))$ where $I : (X,\tau) \to (X,\tau)$ is the identity functions. Since the function I is continuous, then by Theorem 3.13, g is ω_s-continuous. □

Conflic of interest

The authors declare that they have no conflicts of interest to disclse.

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Intermedio entre conjuntos abiertos y semiabiertos

Resumen. Introducimos e investigamos los conjuntos ω_s-abiertos como una nueva clase de conjuntos que se ubica estrictamente entre los conjuntos abiertos y semi-abiertos. Usamos los conjuntos ω_s-abiertos para introducir las funciones ω_s-continuas como un nuevo tipo de funciones que se encuentran entre las funciones continuas y semicontinuas. Proporcionamos varios resultados y ejemplos relacionados con nuestros nuevos conceptos. En particular, obtenemos algunas caracterizaciones de las funciones ω_s-continuas.

Palabras clave: Conjunto semiabierto; Conjunto ω-abierto; Función semicontinua

Intermédio entre conjuntos abertos e semiaberto

Resumo. Introduzimos e investigamos os conjuntos ω_s-abertos como uma nova classe de conjuntos que se localiza estritamente entre os conjuntos abertos e semiabertos. Usamos os conjuntos ω_s-abertos para introduzir as funções ω_s-contínuas como um novo tipo de função que se encontram entre as funções contínuas e semicontínuas. Proporcionamos vários resultados e exemplos relacionados com nossos novos conceitos. Particularmente, obtemos algumas caracterizações das funções ωs-contínuas.

Palavras-chave: Conjunto semiaberto; Conjunto ω-aberto; Função semicontínua

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