

# Neutron Density Calculation Using the Generalised Adams-Bashforth-Moulton Method

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**Received:** 06-06-2018

**Accepted:** 08-07-2019

**Published on line:** 20-11-2019

**Citation:** Suescún-Díaz D, Rasero  
Causil DA, Lozano-Parada JH.  
Neutron Density Calculation Using  
the Generalised Adams-Bashforth-  
Moulton Method,  
*Universitas Scientiarum*, 24 (3): 543-563, 2019.  
doi: [10.11144/Javeriana.SC24-3.ndcu](https://doi.org/10.11144/Javeriana.SC24-3.ndcu)

## Funding:

Applied Physics Research Group of the  
Surcolombian University

**Electronic supplementary material:**

N.A.



## Abstract

This paper presents a numerical solution to the equations of point kinetics for nuclear power reactors, a set of seven coupled differential equations that describe the temporal variation of neutron density and the concentration of delayed neutron precursors. Due to the nature of the system, we propose to numerically solve the point kinetics equations by implementing the Adams-Bashforth and Adams-Moulton methods, which are predictor-corrector schemes with their respective modifiers to increase precision. The proposed method was tested computationally for different forms of reactivity with up to six groups of delayed neutron precursors. This method was used in a recent publication to solve the inverse problem of finding the reactivity. In this work, it is shown that it can also be used for the calculation of nuclear power, that it is simple and easy to implement, and that it produces good results when compared with those in the literature for neutron population density and concentration of delayed neutron precursors.

**Keywords:** nuclear reactor power, nuclear density, point kinetics equations, numerical methods.

## Introduction

A nuclear reactor is a device to initiate, control and maintain nuclear fission reactions. The understanding of nuclear fission processes requires the study of the neutron population and the concentration of delayed neutrons, which can be modelled by the equations of point kinetics. The reactor point kinetic equations are the reduced-order model most representative in the nuclear reactor dynamics.

The calculation of reactivity depends on the neutron density which makes the point kinetics equations nonlinear, and therefore very difficult to solve analytically. The fast and delayed neutron lifetimes are of different orders of magnitude, converting the equations of point kinetics into a stiff system. Several methods have been proposed to solve the equations of point kinetics: In [1], the confinement method (SCM) is proposed to overcome stiffness, whereas other authors have used the generalised Runge-Kutta (GRK) method [2], the Padé approximations [3], the generalization of the analytical inversion method (AIM) [4], and the analytical method (AEM), which used exponential functions [5]. It has also been demonstrated that the point kinetics equations can be solved numerically using the reactivity Piecewise Constant Approximation method (PCA) [6], using a numerical algorithm called: Constant Reactivity (CORE) to calculate nuclear density [7, 8] presented a numerical integral method and investigated the neutron density produced by inserting different forms of reactivity in thermal reactors with multiple groups of the neutrons using the Best Function (BBF). The Taylor series (TSM) was used to calculate the nuclear density with feedback reactivity [9], and the power series method (PWS) has been used to obtain approximate solutions with and without feedback [10-11]. [12] introduced a highly accurate algorithm, combining the Backward Euler Finite Difference method (BEFD) and [13] described a semi-analytical method to solve the equations of point kinetics with a technique called (EPCA) which iteratively corrects the error in the source term with good accuracy. The reduced form of the differential transform method (reduced DT Method) [14] was used to obtain the total neutron density. In [15] the Taylor-Lie series was applied to numerically solve the equations of point kinetics with a quadrature technique. [16] introduced the Haar Wavelet Operational Method (HWOM). In [17] an explicit analytical solution was developed from a Taylor series expansion (ITS2). [18] introduced a new method based on the Trigonometric Fourier Series (TFS) to obtain approximate solutions. In [19] the Exponential Time Differencing Method (ETDM) with a Taylor series approximation was proposed to solve the equations of point kinetics; this is a semi-analytic and self-starting method in which the point kinetic equations are integrated using an integration factor. [20] introduced a matrix form known as Treatment Theta Method (TTM) and in a more recent work the Adams-Bashforth and Adams-Moulton Methods (ABM) were used in nuclear reactivity calculations [21]. Finally, in a recent work [22], neutron density was calculated using the Magnus expansion. This new work proposes to solve the equations of point kinetics for the calculation of nuclear density with the generalised Adams-Bashforth and Adams-Moulton predictor-corrector method of order  $O(h^4)$ , with the objective of studying various predictors using the same corrector. This type of multipass method follows a prediction and correction scheme.

The present work is structured as follows: firstly, a section considering a physical-mathematical model, which contains the theoretical framework and the description of the variables or parameters of the model; secondly, a section with the proposed method, which explains the methodology that was carried out in this research. Finally, a section with the results and conclusions.

### Model formulation

The point kinetics equations describe the time behaviour of the nuclear reactor. These equations are obtained from the equation of neutron diffusion with six groups of delayed neutrons, as given by [23].

$$\frac{dP(t)}{dt} = \left[ \frac{\rho(t) - \beta}{\Lambda} \right] P(t) + \sum_{i=1}^6 \lambda_i C_i(t) \quad (1)$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\Lambda} P(t) - \lambda_i C_i(t) \quad i = 1, 2, \dots, 6 \quad (2)$$

with the following initial conditions:

$$P(t = 0) = P_0 \quad (3)$$

$$C_i(t = 0) = \frac{\beta_i}{\Lambda \lambda_i} P_0 \quad (4)$$

Where  $P(t)$  is neutron density  $C_i(t)$  is the concentration of the  $i$ -th group of delayed neutron precursors,  $\rho(t)$  is the reactivity,  $\Lambda$  is the neutron generation time,  $\beta_i$  is the fraction of the  $i$ -th delayed neutron group  $\beta$  is the total fraction of delayed neutrons, ( $\beta = \sum_i \beta_i$ ),  $\lambda_i$  is the decay constant of the  $i$ -th group of delayed neutron precursors.

### Adams-Bashforth and Adams-Moulton Methods

The Adams-Bashforth-Moulton multistep method (ABM) solves the initial value problem of the form  $y' = f(t_k, y_k)$  using information from points previous to  $y_k$  to find  $y_{k+1}$ , generally, these points are obtained with the Runge-Kutta method of order  $O(h^4)$ . The method of indeterminate coefficients is used to calculate the solution of the function  $f(t_k, y_k)$ , the differential equation with indeterminate coefficients for the generalised fourth-order predictor of the Adams-Bashforth method can be represented by the following equation:

$$P_{k+1} = Y_{k+1} = \sum_{i=1}^4 \alpha_i Y_{k+1-i} + h \sum_{i=1}^4 \beta_i y'_{k+1-i} + E_p \quad (5)$$

The coefficients  $\alpha_i, \beta_i$  for  $i = 1, \dots, 4$  are calculated by carrying out a Taylor series expansion around  $h$  on each term  $y_{k-1}, y_{k-2}, y_{k-3}, y'_{k-1}, y'_{k-2}, y'_{k-3}$ , of Eq. (5). We have that

$$Y_{k-1} = y_k + y'_k(-ih) + \left(\frac{1}{2!}\right)y_k^{(2)}(-ih)^2 + \left(\frac{1}{3!}\right)y_k^{(3)}(-ih)^3 + \left(\frac{1}{4!}\right)y_k^{(4)}(-ih)^4 + \left(\frac{1}{5!}\right)y_k^{(5)}(\xi)(-ih)^5 \quad (6)$$

$$Y'_{k-1} = y'_k + y_k^{(2)}(-ih) + \left(\frac{1}{2!}\right)y_k^{(3)}(-ih)^2 + \left(\frac{1}{3!}\right)y_k^{(4)}(-ih)^3 + \left(\frac{1}{4!}\right)y_k^{(5)}(\xi)(-ih)^4 \quad (7)$$

Replacing Eqs. (6, 7) into Eq. (5) and equating to the Taylor series for  $Y_{k+1}$ , that is, making  $i = -1$  in Eq. (6), we obtain a set of algebraic equations

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 1 \\ -\alpha_2 - 2\alpha_3 - 3\alpha_4 + \beta_1 + \beta_2 + \beta_3 + \beta_4 &= 1 \\ \frac{1}{2}\alpha_2 + 2\alpha_3 + \frac{9}{2}\alpha_4 - \beta_2 - 2\beta_3 - 3\beta_4 &= \frac{1}{2} \\ -\frac{1}{6}\alpha_2 - \frac{4}{3}\alpha_3 - \frac{9}{2}\alpha_4 + \frac{1}{2}\beta_2 + 2\beta_3 + \frac{9}{2}\beta_4 &= \frac{1}{6} \\ \frac{1}{24}\alpha_2 + \frac{2}{3}\alpha_3 + \frac{27}{8}\alpha_4 - \frac{1}{6}\beta_2 - \frac{4}{3}\beta_3 - \frac{9}{2}\beta_4 &= \frac{1}{24} \end{aligned} \quad (8)$$

By solving Eqs. (8) we obtain the coefficients/predictors. **Table 1** shows these values, which are then replaced into Eq. (5), thus, different predictor formulas can be obtained. For the error estimation, it must be taken into account that

$$E_p = \frac{E}{5!} h^5 y_k^{(5)}(\xi).$$

Similarly, the corrector of the Adams-Moulton method can be obtained or deduced from the fundamental theorem of calculus, and the Lagrange's interpolator polynomial is given by:

**Table 1.** Coefficients of the generalised predictor.

Coefficientes \ Predictores	P1	P2	P3	P4	P5	P6
$\alpha_1 = 1 - \alpha_2 - \alpha_3 - \alpha_4$	1	0	0	0	0	1/2
$\alpha_2 = \alpha_2$	0	1	0	0	1/3	1/2
$\alpha_3 = \alpha_3$	0	0	1	0	1/3	0
$\alpha_4 = \alpha_4$	0	0	0	1	1/3	0
$\beta_1 = (1/24) (55 + 9\alpha_2 + 8\alpha_3 + 9\alpha_4)$	55/24	8/3	21/8	8/3	191/72	119/48
$\beta_2 = (1/24) (-59 + 19\alpha_2 + 32\alpha_3 + 27\alpha_4)$	-59/24	-5/3	-9/8	-4/3	-11/8	-99/48
$\beta_3 = (1/24) (37 + 5\alpha_2 + 8\alpha_3 + 27\alpha_4)$	37/24	4/3	15/8	8/3	47/24	69/48
$\beta_4 = (1/24) (-9 + \alpha_2 + 9\alpha_4)$	-9/24	-1/3	-3/8	0	-17/72	-17/48
$E = (1/6) (251 - 19\alpha_2 - 8\alpha_3 - 27\alpha_4)$	251/6	116/3	243/6	224/6	233/6	116/4

$$Y_{k+1} = y_k + \frac{h}{24}(9y'_{k+1} + 19y'_k - 5y'_{k-1} + y'_{k-2}) \tag{9}$$

The truncation error of Eq. (9) is:

$$E_c = -\frac{19}{720}h^5 y_k^{(5)}(\xi) \tag{10}$$

The method's convergence criterion is obtained by making successive adjustments to the corrector in the following way:

$$\begin{aligned}
 y_\alpha - y_c &= \left( y_k + \frac{h}{24}(9y'_c + 19y'_k - 5y'_{k-1} + y'_{k-2}) \right) \\
 &\quad - \left( y_k + \frac{h}{24}(9y'_p + 19y'_k - 5y'_{k-1} + y'_{k-2}) \right) \\
 &= \frac{9h}{24}(y'_c - y'_p)
 \end{aligned} \tag{11}$$

where,

$y_p, y_c$  are values in the predictor-corrector formulas,  $y_{cc}$  is the first correction in the corrector

After some algebraic manipulation, Eq. (11) can be written as:

$$y_{cc} - y_c = \frac{9hD}{24} f_y(\xi) \quad (12)$$

where,  $D = y_c - y_p$ ,  $\xi \in [y_c, y_p]$ . After a new correction, the result will be:

$$y_{ccc} - y_{cc} = \frac{9h}{24} (y'_{cc} - y'_c) = \left(\frac{9h}{24}\right)^2 [f_y(\xi)]^2 D \quad (13)$$

Thus, by adding all the corrections, a geometric series with ratio  $r$  arises

$$r = \frac{9hf_y(\xi)}{24} \quad (14)$$

In order for the geometric series as given by Eq. (14) to converge, we must have  $r < 1$ . Rearranging for  $h$  gives the convergence criteria as

$$h \leq \frac{24}{9|f_y(t, y)|} \quad (15)$$

The case predictor proposed (P3), is selected from Table 1, obtaining:

$$p_{k+1} = y_{k-2} + \frac{h}{8} (21y'_k - 9y'_{k-1} + 15y'_{k-2} - 3y'_{k-3}) \quad (16)$$

The predictor error of Eq. (16) is given by:

$$E_p = \frac{243}{720} h^5 y_k^{(5)}(\xi) \quad (17)$$

In order to obtain better results, we can find the corrector's modifier and the predictor in the following way

$$y(t_{k+1}) = p_{k+1} + E_p = y_{k+1} + E_c \quad (18)$$

where  $y(t_{k+1})$  is the exact value.

Replacing Eq. (10) and Eq. (17) into Eq. (18), we get:

$$p_{k+1} - y_{k+1} = E_c - E_p = -\frac{19}{720} h^5 y_k^{(5)}(\xi) - \frac{243}{720} h^5 y_k^{(5)}(\xi) = -\frac{262}{720} h^5 y_k^{(5)}(\xi) \quad (19)$$

The term that contains the derivative can be expressed as

$$\frac{720}{262}(p_{k+1} - y_{k+1}) = -h^5 y_k^{(5)} \xi \quad (20)$$

Multiplying Eq. (20) by  $\frac{19}{720}$ , and making use of Eq. (10) we get

$$\frac{19}{262}(p_{k+1} - y_{k+1}) = -\frac{19}{720} h^5 y_k^{(5)}(\xi) = E_c \quad (21)$$

By replacing Eq. (21) into Eq. (18) we get

$$y(t_{k+1}) = y_{k+1} + e_c = y_{k+1} + \frac{19}{262}(P_{k+1} - y_{k+1}) \quad (22)$$

Adding  $-p_{k+1}$  to both sides of Eq. (22) we get

$$y(t_{k+1}) \approx P_{k+1} + \frac{243}{262}(y_{k+1} - P_{k+1}) \quad (23)$$

Eq. (23) can be used to improve the value of the predictor and to estimate the error both in the predictor and the corrector, without having to calculate the fifth derivative. A slow change in the difference between the predictor and corrector from one step to the next is assumed. Then  $P_{k+1}$  is replaced by  $P_k$ ,  $y_{k+1}$  by  $y_k$  in Eq. (23) to obtain the following modified formula for the predictor:

$$m p_{k+1} = p_{k+1} + \frac{243}{262}(y_k - p_k) \quad (24)$$

Similarly, the modified formula for the corrector is:

$$m y_{k+1} = y_{k+1} - \frac{19}{262}(y_{k+1} - p_{k+1}) \quad (25)$$

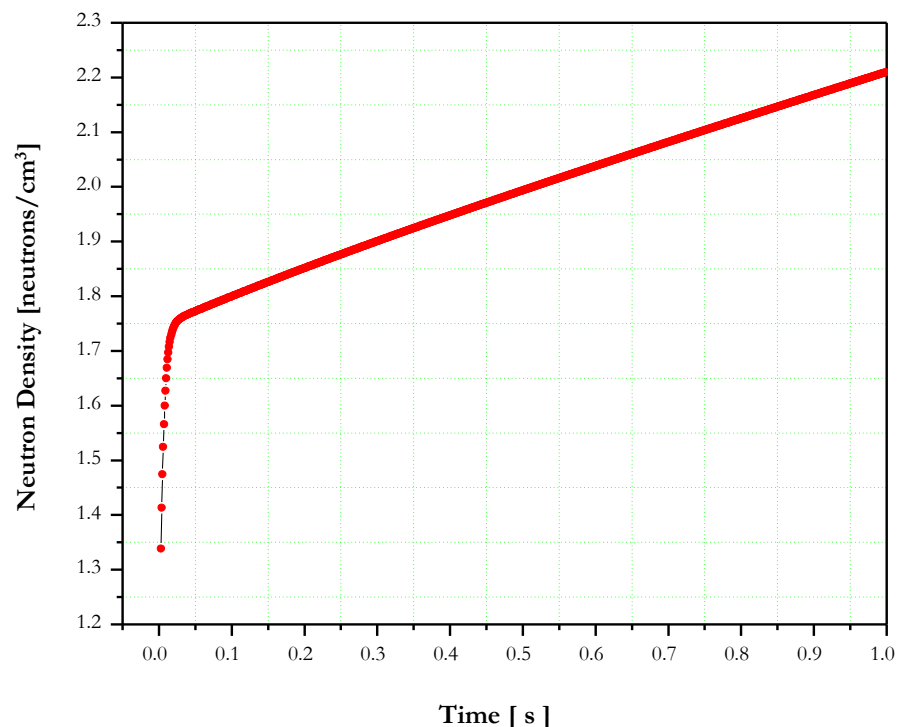
## Results and discussion

The Adams-Bashforth-Moulton method is applied to solve the equations of point kinetics with one or six groups of delayed neutrons. Three cases of reactivity functions are presented: step, ramp and sinusoidal. Each case will be studied separately and the results will be compared to other methods available in the literature.

## Step Reactivity

For the neutron density calculation the following parameters for a thermal reactor are considered:  $\lambda_1 = 0.0127 \text{ s}^{-1}$ ,  $\lambda_2 = 0.0317 \text{ s}^{-1}$ ,  $\lambda_3 = 0.115 \text{ s}^{-1}$ ,  $\lambda_4 = 0.311 \text{ s}^{-1}$ ,  $\lambda_5 = 1.4 \text{ s}^{-1}$ ,  $\lambda_6 = 3.87 \text{ s}^{-1}$ ,  $\beta_1 = 0.0000266$ ,  $\beta_2 = 0.001491$ ,  $\beta_3 = 0.001316$ ,  $\beta_4 = 0.002848$ ,  $\beta_5 = 0.000896$ ,  $\beta_6 = 0.000182$ ,  $\beta = 0.007$ ,  $\Lambda = 2 \times 10^{-5} \text{ s}$ .

**Table 2** and **Table 3** show different values for neutron density with reactivities of 300 pcm = 0.43 (\$) and 700 pcm (parts per hundred thousand) = 1 (\$) reported in the literature [17], these are values for critical and subcritical reactors. The dollar symbol 1 (\$) is equivalent to  $1 \beta = 0.007$ . The step size in the proposed ABM method is  $h = 10^{-3} \text{ s}$ , unfortunately, the authors do not show the time step used in their reported methods. **Fig. 1** shows how stiffness forces an abrupt change in the neutron density for a reactivity of 300 pcm = 0.43 (\$). This is due to the big differences between  $\Lambda = 5 \times 10^{-4} \text{ s}$  and  $1/\lambda_i$ .



**Figure 1.** Neutron density for reactivity  $\rho = 0.43$  (\$).



**Table 2.** Neutron density for reactivity  $\rho = 0.43$  (\$).

t(s)	0.4	0.6	0.8	1.0
BBF	1.947 585	2.037 917	2.124 829	2.209 839
Taylor	1.947 581	2.037 915	2.124 827	2.209 838
End Floating	1.947 586	2.037 918	2.124 829	2.209 839
ITS2	1.947 593 411 38	2.037 922 055 92	2.124 831 636 42	2.209 840 456 98
ABM	1.947 593 411 22	2.037 922 055 78	2.124 831 636 28	2.209 840 456 84

The ABM method can be applied to a thermal reactor with the values obtained in Yun Cai *et al.* [22] using the ME3, PCA / ME2 and EPCA methods. The values of the kinetic parameters are:  $\lambda_1 = 0.0127 \text{ s}^{-1}$ ,  $\lambda_2 = 0.0317 \text{ s}^{-1}$ ,  $\lambda_3 = 0.115 \text{ s}^{-1}$ ,  $\lambda_4 = 0.311 \text{ s}^{-1}$ ,  $\lambda_5 = 1.4 \text{ s}^{-1}$ ,  $\lambda_6 = 3.87 \text{ s}^{-1}$ ,  $\beta_1 = 0.000285$ ,  $\beta_2 = 0.0015975$ ,  $\beta_3 = 0.00141$ ,  $\beta_4 = 0.0030535$ ,  $\beta_5 = 0.00096$ ,  $\beta_6 = 0.000195$ ,  $\beta = 0.0075$ ,  $\Lambda = 5 \times 10^{-4} \text{ s}$ . **Table 4** shows the results obtained with the ABM method and those reported in the literature for a reactivity  $\rho = 1$  (\$).

### Ramp reactivity

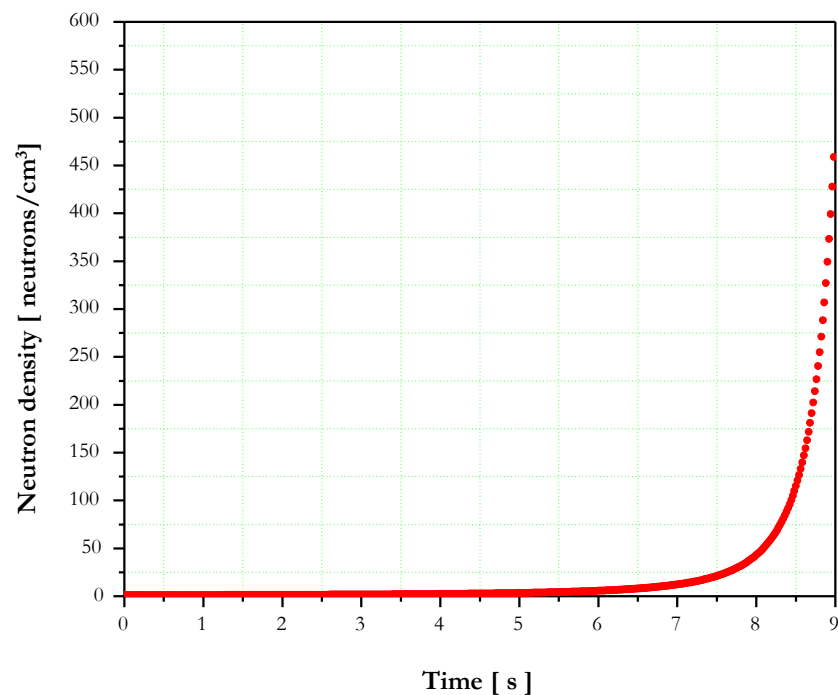
In this section, the behaviour of the neutron density is determined for ramp reactivities of two types: moderately fast and very fast, which correspond to thermal and fast reactors, respectively.

In the case of moderately fast ramp reactivities, the parameters for the thermal reactor are obtained from [17]:  $\lambda_1 = 0.0127 \text{ s}^{-1}$ ,  $\lambda_2 = 0.0317 \text{ s}^{-1}$ ,  $\lambda_3 = 0.115 \text{ s}^{-1}$ ,  $\lambda_4 = 0.311 \text{ s}^{-1}$ ,  $\lambda_5 = 1.4 \text{ s}^{-1}$ ,  $\lambda_6 = 3.87 \text{ s}^{-1}$ ,  $\beta_1 = 0.000266$ ,  $\beta_2 = 0.001491$ ,  $\beta_3 = 0.001316$ ,  $\beta_4 = 0.002848$ ,  $\beta_5 = 0.000896$ ,  $\beta_6 = 0.000182$ ,  $\beta = 0.007$ ,  $\Lambda = 2 \times 10^{-5} \text{ s}$ . **Table 5** and **Table 6** show the calculation of neutron density given a reactivity of the form  $\rho(t) = 0.0007 \cdot t$ , in this case  $\rho = 0.1$ (\$)/s. The results using the proposed ABM method are compared with the values reported in references [17] and [22]. The proposed method uses the step size  $h = 10^{-3} \text{ s}$  again. It is observed that the ABM method is very accurate when compared to other methods in the literature. For this type of reactivity, the neutron density increases exponentially in such a way that for a time  $t = 9$ , there are approximately 487 neutrons /  $\text{cm}^3$ , as can be observed in **Fig. 2**.

**Table 3.** Neutron density for reactivity  $\rho = 1$  (\$).

t(s)	0.4	0.6	0.8	1.0
BBF	0.465 290 4	0.451 964 8	0.440 272 8	0.429.782 5
Taylor	0.465 290 9	0.451 965 1	0.440 273 1	0.429.782 7
End Floating	0.465 290 4	0.451 964 7	0.440 272 7	0.429.782 4
ITS2	0.465 289 326 117	0.451 963 975 793	0.440 272 277 652	0.429.782 046 265
ABM	0.465 289 326 298	0.451 963 975 950	0.440 272 277 791	0.429.782 046 391

In the case of very fast ramp reactivities, the ABM method can be applied to a fast reactor with the parameters obtained in Picca [13] using the EPCA method, and from Yun Cai *et al.* [22] using the ME3 and PCA/ME2 methods. The values of the kinetic parameters are:  $\lambda_1 = 0.0129 \text{ s}^{-1}$ ,  $\lambda_2 = 0.0311 \text{ s}^{-1}$ ,  $\lambda_3 = 0.134 \text{ s}^{-1}$ ,  $\lambda_4 = 0.331 \text{ s}^{-1}$ ,

**Figure 2.** Neutron density obtained with a reactivity of  $\rho = 0.1$  (\$).

**Table 4.** Neutron density for reactivity  $\rho = 1$  (\$).

t(s)	PCA/ME2	ME3	BEFD
0.1	2.515 766 141e+0	2.515 766 141e+0	2.515 766 141e+0
0.5	1.036 253 381e+1	1.036 253 381e+1	1.036 253 381e+1
1.0	3.218 354 095e+1	3.218 354 095e+1	3.218 354 095e+01
10	3.246 978 898e+9	3.246 978 898e+9	3.246 978 898e+09
100	2.596 484 647e+89	2.596 484 647e+89	2.596 484 647e+89

t(s)	ABM (h = 0.1 s)	ABM (h = 0.01 s)	ABM (h = 0.001 s)
0.1	2.515 682 234e+0	2.515 766 144e+0	2.515 766 141e+0
0.5	1.036 240 672e+1	1.036 253 383e+1	1.036 253 381e+1
1.0	3.218 326 024e+1	3.218 354 105e+1	3.218 354 095e+1
10	3.247 549 087e+1	3.246 979 019e+9	3.246 978 898e+9
100	2.601 738 443e+89	2.596 485 635e+89	2.596 484 647e+89

$\lambda_5 = 1.26 \text{ s}^{-1}$ ,  $\lambda_6 = 3.21 \text{ s}^{-1}$ ,  $\beta_1 = 0.0001672$ ,  $\beta_2 = 0.001232$ ,  $\beta_3 = 0.0009504$ ,  $\beta_4 = 0.001443$ ,  $\beta_5 = 0.0004534$ ,  $\beta_6 = 0.0001541$ ,  $\beta = 0.00440$ ,  $\Lambda = 1 \times 10^{-7} \text{ s}$ . **Table 7** shows the results obtained with the ABM method and those reported in the literature for a reactivity  $\rho = 1$  (\$)/s.

### Sinusoidal Reactivity

In order to validate the ABM approach, the results obtained for the sinusoidal reactivity are compared with the results obtained in [13] and [22] using the ME3, PCA / ME2 and EPCA methods with nonlinear reactivity insertion for a fast reactor with the following parameters: where  $\lambda_1 = 0.077 \text{ s}^{-1}$ ,  $\beta = \beta_1 = 0.0079$ ,  $\Lambda = 1 \times 10^{-6} \text{ s}$ . The results are shown in **Table 8**.

Another validation for the approximation of the ABM method consists in comparing it with the results obtained in [19] using the ETD and TSM methods with nonlinear reactivity insertion for a thermal reactor with the following parameters:  $\lambda_1 = 0.0127 \text{ s}^{-1}$ ,  $\lambda_2 = 0.0317 \text{ s}^{-1}$ ,  $\lambda_3 = 0.115 \text{ s}^{-1}$ ,

**Table 5.** Neutron density obtained with a reactivity of  $\rho = 0.1$  (\$)/s.

t(s)	ITS2	GRK (h = 0.056 s)	pws (h = 0.01 s)
2.0	1.338 200 050e+0	1.338 2 e+0	1.338 2 e+0
4.0	2.228 441 897e+0	2.228 4 e+0	2.228 4 e+0
6.0	5.582 052 449e+0	5.582 0 e+0	5.582 0 e+0
8.0	4.278 629 573e+1	4.278 6 e+1	4.278 6 e+1
10.0	4.511 636 239e+5	4.511 5 e+5	4.511 6 e+5
11.0	1.792 213 608e+16	1.790 4 e+16	1.792 1 e+16

t(s)	⊖-Weighting (h = 0.0001)	AIM (h = 0.001 s)	ABM (h = 0.001 s)
2.0	1.338 2 e+0	1.338 2 e+0	1.338 200 011 e+0
4.0	2.228 3 e+0	2.228 4 e+0	2.228 441 842 e+0
6.0	5.581 5 e+0	5.582 0 e+0	5.582 052 323 e+0
8.0	4.278 1 e+1	4.278 6 e+1	4.278 629 479 e+1
10.0	4.510 9 e+5	4.511 6 e+5	4.511 636 145 e+5
11.0	1.791 9 e+16	1.791 6 e+16	1.792 213 979 e+16

$\lambda_4 = 0.311 \text{ s}^{-1}$ ,  $\lambda_5 = 1.4 \text{ s}^{-1}$ ,  $\lambda_6 = 3.87 \text{ s}^{-1}$ ,  $\beta_1 = 0.000266$ ,  $\beta_2 = 0.001491$ ,  $\beta_3 = 0.001316$ ,  $\beta_4 = 0.002848$ ,  $\beta_5 = 0.000896$ ,  $\beta_6 = 0.000182$ ,  $\beta = 0.007$ ,  $\Lambda = 2 \times 10^{-5} \text{ s}$ ,  $\rho(t) = \rho_o \text{ Sin}(4\pi t)$  where  $\rho_o = 0.001$ . These results are shown in **Table 9**. This form of inserted reactivity produces an oscillation in the neutron population density, as can be seen in **Fig. 3**.

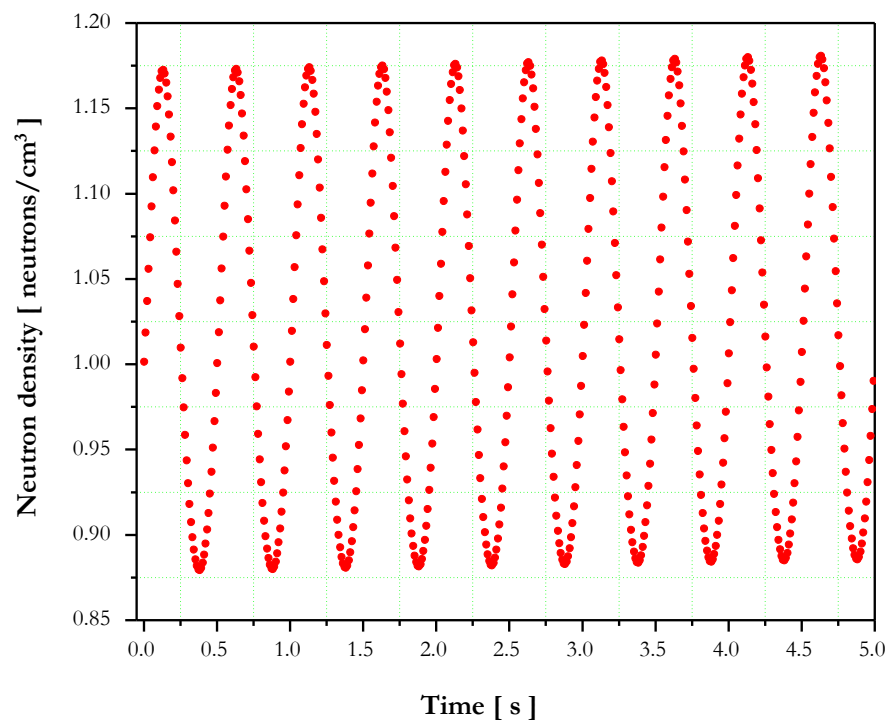
The results presented using the proposed method of ABM of order 4 can be summarised in the following way: using a time step  $h = 0.001\text{s}$ , for thermal reactors with Step Reactivity, at least nine significant figures could be obtained. For Ramp Reactivity for thermal and fast reactors, at least six significant figures are obtained in the numerical experiments. Finally, three significant figures are obtained for experiments with Sinusoidal Reactivity for thermal and fast reactors. For all the experiments performed in this work, better results can be achieved with a time step  $h = 0.0001\text{s}$ .

**Table 6.** Neutron density obtained with a reactivity of  $\rho = 0.43$  (\$).

t(s)	ME3 (h = 0.001 s)	PCA/ME2 (h = 0.001 s)	TSM
2.0	1.338 200 055 e+0	1.338 196 154 e+0	1.338 2 e+0
4.0	2.228 441 902 e+0	2.228 435 405 e+0	2.228 4 e+0
6.0	5.582 052 454 e+0	5.582 036 180 e+0	5.582 2 e+0
8.0	4.278 629 574 e+1	4.278 617 100 e+1	4.278 9 e+1
10.0	4.511 636 239 e+5	4.511 623 086 e+5	4.514 3 e+5
11.0	1.792 213 607 e+16	1.792 208 382 e+16	NA

t(s)	BBF	BEFD	ABM (h = 0.001 s)
2.0	1.338 2 e+0	1.338 200 050 e+0	1.338 200 011 e+0
4.0	2.228 4 e+0	2.228 441 897 e+0	2.228 441 842 e+0
6.0	5.582 0 e+0	5.582 052 449 e+0	5.582 052 323 e+0
8.0	4.278 6 e+1	4.278 629 573 e+1	4.278 629 479 e+1
10.0	4.504 1 e+5	4.511 636 239 e+5	4.511 636 145 e+5
11.0	NA	1.792 213 607 e+16	1.792 213 979 e+16



**Figure 3.** Neutron density for a thermal reactor with sinusoidal reactivity.

**Table 7.** Neutron density for reactivity  $\rho = 1$  (\$).

<b>t(s)</b>	<b>PCA/ME2</b> <b>(h = 0.0001 s)</b>	<b>ME3</b> <b>(h = 0.0001 s)</b>	<b>EPCA</b> <b>(h = 0.00001 s)</b>	<b>ABM</b> <b>(h = 0.00001 s)</b>
0.01	1.010 068 194e+0	1.010 105 230e+0	1.010 097 112e+0	1.010 097 111e+0
0.1	1.113 287 096e+0	1.113 327 917e+0	1.113 320 113e+0	1.113 320 112e+0
0.2	1.260 521 100e+0	1.260 567 318e+0	1.260 559 925e+0	1.260 559 925e+0
0.5	2.136 336 445e+0	2.136 414 776e+0	2.136 409 108e+0	2.136 409 107e+0
1.0	1.207 769 913e+3	1.207 814 194e+3	1.207 814 198e+3	1.207 814 197e+3
1.1	3.257 473 916e+99	3.257 593 342e+99	3.257 593 356e+99	3.257 601 607e+99
1.15	1.028 937 632e+219	1.028 975 355e+219	1.028 975 360e+219	1.029 000 660e+219

**Table 8.** Neutron density for a fast reactor with sinusoidal reactivity.

<b>t(s)</b>	<b>ME3</b> <b>(h = 0.001 s)</b>	<b>PCA/ME2</b> <b>(h = 0.001 s)</b>	<b>EPCA</b> <b>(h = 0.0001 s)</b>	<b>ABM</b> <b>(h = 0.0001 s)</b>
10	2.065 322 671e+0	2.065 276 012e+0	2.065 311 114e+0	2.065 311 114e+0
20	8.852 839 730e+0	8.852 763 321e+0	8.852 831 171e+0	8.852 831 171e+0
30	4.063 324 406e+1	4.063 359 451e+1	4.063 328 342e+1	4.063 328 342e+1
40	6.134 100 265e+1	6.134 238 819e+1	6.134 134 599e+1	6.134 134 599e+1
50	4.609 503 985e+1	4.609 632 692e+1	4.609 559 630e+1	4.609 559 630e+1

**Table 9.** Neutron density for a thermal reactor with sinusoidal reactivity.

t(s)	Modified ETD (S = 7) h = 0.001 s	TSM (h = 0.001 s)	ABM (h = 0.001 s)
0.0	1.000 000 0	1.000 000 0	1.000 000 0
0.4	0.882 848 8	0.882 856 4	0.882 516 3
0.8	0.933 438 7	0.933 366 6	0.934 690 9
1.2	1.107 011 5	1.106 932 9	1.108 608 6
1.6	1.161 529 7	1.161 559 7	1.160 647 8
2.0	0.999 293 9	0.999 377 4	0.997 604 7
2.4	0.885 733 0	0.885 750 6	0.885 409 3
2.8	0.936 697 9	0.936 635 7	0.937 954 6
3.2	1.110 873 7	1.110 794 9	1.112 474 9
3.6	1.165 361 6	1.165 401 9	1.164 484 8
4.0	1.002 481 1	1.002 564 9	1.000 786 3
4.4	0.888 574 8	0.888 592 5	0.888 250 7
4.8	0.939 734 6	0.939 672 1	0.941 005 7
5.0	1.003 961 2	1.004 055 1	1.002 263 9

## Conclusions

In this paper, the generalised predictor of the Adams-Bashforth-Moulton method (ABM) was presented. This is a method based upon prediction and correction, that guarantees a higher precision when numerically solving a system of differential equations, in this case applied to the point kinetics equations. The results were presented for case P3 of Table 1, since the other cases had a similar approximation. Different time steps were used to perform the numerical simulations; however, the best results were obtained for  $h = 10^{-3}$ , which allowed the comparison with the results reported in the literature. The values of nuclear power obtained with the proposed method agree very well with those found in the literature for different forms of reactivity, for both thermal and fast reactors. It is concluded that the method can be used to calculate nuclear density under different values of reactivity.

## Acknowledgements

This work was supported by the Applied Physics Research Group of the Surcolombian University, Neiva, Colombia.

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## Cálculo de densidad de neutrones utilizando el método generalizado de Adams-Bashforth-Moulton

**Resumen:** Este artículo presenta una solución numérica a las ecuaciones de cinética puntual para reactores de energía nuclear, un conjunto de siete ecuaciones diferenciales acopladas que describen la variación temporal de la densidad de neutrones y la concentración de precursores de neutrones retardados. Debido a la naturaleza del sistema, proponemos resolver numéricamente las ecuaciones de cinética de puntos mediante la implementación de los métodos de Adams-Bashforth y de Adams-Moulton, que son esquemas predictores-correctores con sus respectivos modificadores para aumentar la precisión. El método propuesto se probó computacionalmente para diferentes formas de reactividad con hasta seis grupos de precursores de neutrones retardados. Este método se utilizó en una publicación reciente para resolver el problema inverso de encontrar la reactividad. Adicionalmente, se muestra que también se puede utilizar para el cálculo de la energía nuclear, que es simple y fácil de implementar, y que produce buenos resultados en comparación con los de la literatura para la densidad de población de neutrones y la concentración de precursores de neutrones retardados.

**Palabras clave:** densidad nuclear; potencia de reactor nuclear; métodos numéricos; ecuaciones de la cinética puntual.

## Cálculo da densidade de nêutrons usando o método generalizado de Adams-Bashforth-Moulton

**Resumo:** Este artigo apresenta uma solução numérica para as equações da cinética pontual para reatores de energia nuclear, um conjunto de sete equações diferenciais acopladas que descrevem a variação temporal da densidade de nêutrons e concentração de precursores de nêutrons atrasados. Devido à natureza do sistema, propomos resolver numericamente as equações da cinética pontual implementando os métodos de Adams-Bashforth e de Adams-Moulton, que são esquemas preditores-corretores com seus respectivos modificadores para aumentar a precisão. O método proposto foi testado computacionalmente para diferentes formas da reatividade com até seis grupos de precursores de nêutrons atrasados. Este método foi usado em uma publicação recente para resolver o problema inverso de encontrar a reatividade. Além disso, mostra-se que também pode ser utilizado para o cálculo da potência nuclear, que é simples e fácil de implementar e que produz bons resultados quando comparado com os da literatura para densidade populacional de nêutrons e concentração de precursores de nêutrons atrasados.

**Palavras-chave:** densidade nuclear; potência do reator nuclear; métodos numéricos; equações da cinética pontual.

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