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ORIGINAL ARTICLE

Existence of local and global solution for a spatio-temporal predator-prey model

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Abstract

In this paper we prove the existence and uniqueness of weak solutions for a kind of Lotka–Volterra system, by using successive linearization techniques. This approach has the advantage to treat two equations separately in each iteration step. Under suitable initial conditions, we construct an invariant region to show the global existence in time of solutions for the system. By means of Sobolev embeddings and regularity results, we find estimates for predator and prey populations in adequate norms. In order to demonstrate the convergence properties of the introduced method, several numerical examples are given.

Keywords: global weak solution; iterative method; predator-prey system.

Introduction

This work is concerned with a special evolution variant of the predator-prey system with homogeneous Neumann boundary conditions

$$\begin{cases} \left.\begin{aligned} u_t - d_1\Delta u = f(u,v) \\ v_t - d_2\Delta v = g(u,v) \end{aligned}\right\} \text{ in } \Omega_T, \\ \dfrac{\partial u}{\partial \eta} = 0,\ \dfrac{\partial v}{\partial \eta} = 0 \text{ on } \partial\Omega \times [0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \text{ on } \Omega, \end{cases} \tag{1}$$

where Ω is an open, bounded subset of $\mathbb{R}^n$, and $\Omega_T := \Omega \times [0,T)$.

The functions $f(u,v)$ and $g(u,v)$ describe the interaction between the prey and predator densities u and v respectively. These are given by

$$f(u,v) = u(a_1 - b_1 u) - c_1 uv \tag{2}$$

and

$$g(u,v) = c_2 uv - a_2 v. \tag{3}$$

The parameters d_1 and d_2 measure the tendency of each population to spread, a_1 is the birth rate of prey, a_2 is the death rate of the predators, b_1 is the decay rate of preys due to the competition among themselves for limited resources, and c_1 and c_2 are measures of the effect of the interaction between the two species. The maximal concentration of preys in the absence of predators (i.e. the carrying capacity of the environment) is $\gamma := a_1/b_1$. We assume that all these parameters are positive.

The interaction of predator and prey populations with logistic growth of prey, without considering any spatial variations in populations density, has been well studied in [1] and [2]. Concerning the solvability of the spatially extended predator-prey system (1) several techniques have been proposed. In [3] and [4] the existence of traveling wave solutions is studied and some possible biological implications are given. In [5], the authors employ the implicit function theorem and spectral theory to show existence and uniqueness of the positive steady-state solutions of a reaction-diffusion two-competition species model with advection term under the Dirichlet boundary condition. In [6], it was established the existence of a solution for a fractional differential Lotka–Volterra reaction-diffusion equation. The technique of upper and lower solutions is used in [7], to show the existence and uniqueness of a classical global time-dependent solution and its asymptotic relation with the steady-state solutions. Another line of research is presented in [8], where the global existence and boundedness of classical solutions is studied for a predator-prey model via semigroup theory. The existence and uniqueness of weak solutions of the system (1) in $\Omega \subset \mathbb{R}^n$ with homogeneous Dirichlet boundary conditions have been studied in [9] using the semi-implicit Rothe method.

In this work, we use a different strategy in order to obtain the existence and uniqueness of a weak solution for system (1). We propose an iterative process, which solves at each iteration a linear problem. For each linear problem, we prove the existence and uniqueness of a weak solution. We obtain estimates for the iterative sequence. In the next step, we show that this sequence of solutions of linear problems is a Cauchy sequence in an appropriate Banach space, and consequently, converges. From this convergence, the weak solution for the full original nonlinear system is obtained. Furthermore, the global existence of solutions for system (1) with suitable initial conditions is proved via the technique of invariant region.

The main virtue of this iterative approach is that due to the structure of the resulting linear problems, it is easy to prove the existence and uniqueness of a weak solution as well as to obtain a priori uniform estimates for the generated sequence. Another virtue is that the technique developed here could be applied to more complex nonlinear problems and adapted to a numerical scheme that solves these kinds of problems.

Methods

We show for a fixed time $T > 0$, the system (1) has exactly one weak solution. We will formulate the nonlinear system (1) as a decoupled linear system in a weaker space setting, and show that a weak solution exists as the limit of solutions to corresponding approximation systems. In addition, we use embedding theorems and classical regularity results to obtain estimates for the predator and prey densities.

Definition 2.1. A pair (u,v) of functions in $L^2\big(0,T;H^1(\Omega)\big) \times L^2\big(0,T;H^1(\Omega)\big)$ with $u_t, v_t \in L^2\big(0,T;H^1(\Omega)'\big)$ is a weak solution of system (1), if both equations

$$\int_\Omega u_t\,\phi\,dx + \int_\Omega d_1 \nabla u\,\nabla\phi\,dx = \int_\Omega f(u,v)\,\phi\,dx, \tag{4}$$

$$\int_\Omega v_t\,\phi\,dx + \int_\Omega d_2 \nabla v\,\nabla\phi\,dx = \int_\Omega g(u,v)\,\phi\,dx, \tag{5}$$

are satisfied for all $\phi \in H^1(\Omega)$, a.e. in $[0,T]$.

The most important result of this paper is the following theorem.

Theorem 2.2. *Let $0 \leqslant \rho$ and $\gamma \leqslant a_2/c_2$ with $\gamma = a_1/b_1$. If $u_0, v_0 \in H^1(\Omega)$ such that*

$$0 \leqslant u_0 \leqslant \gamma \quad and \quad 0 \leqslant v_0 \leqslant \rho.$$

Then, there exists a unique solution (u,v) of the weak formulations (4)–(5) for some $T > 0$.

In order to prove Theorem 2.2 we will use frequently the following well-known result of linear partial differential equations.

Theorem 2.3. *(See [10], Theorems 1–4, §7.1.2 and 5, §7.1.3)*
Assume $a(x,t),\ c(x,t) \in L^\infty(\Omega_T)$. *Then for any* $u_0(x) \in H^1(\Omega)$ *and* $f(x,t) \in L^2(0,T;L^2(\Omega))$, *there exists a unique weak solution* $u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$ *with* $u_t \in L^2(0,T;H^{-1}(\Omega))$ *to the problem*

$$\begin{cases} u_t - \nabla\cdot\big(a(x,t)\nabla u\big) + c(x,t)u = f(x,t) & \text{in } \Omega_T, \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \\ u = u_0 & \text{on } \Omega\times\{t=0\}. \end{cases}$$

Moreover,

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T} ||u(t)||_{H^1(\Omega)} + ||u||_{L^2(0,T;H^2(\Omega))} + ||u_t||_{L^2(0,T;L^2(\Omega))} \leqslant C\left(||u_0||_{H^1(\Omega)} + ||f||_{L^2(0,T;L^2(\Omega))}\right) \quad (6)$$

with $C = C(\Omega, T, ||a||_{L^\infty(\Omega_T)}, ||c||_{L^\infty(\Omega_T)})$.

The strategy of the proof is: to construct a sequence of linear approximations to the problem (1); in this way, we reduce the nonlinear coupled system (1) into a sequence of linear systems (9)–(10). After that, we establish that the resulting solutions (u^k, v^k) define a Cauchy sequence in a suitable Banach space, which converges to a solution of our problem.

Iterative sequence: Let $(u^0, v^0) \in L^2\big(0,T;H^2(\Omega)\big) \times L^2\big(0,T;H^2(\Omega)\big)$ be a weak solution of (7) and (8), with $u_t^0, v_t^0 \in L^2\big(0,T;L^2(\Omega)\big)$

$$\begin{cases} u_t^0 - d_1\Delta u^0 = 0 & \text{in } \Omega_T, \\ \frac{\partial u^0}{\partial \eta} = 0 & \text{on } \partial\Omega\times[0,T), \\ u^0(x,0) = u_0(x) & \text{on } \Omega; \end{cases} \quad (7)$$

$$\begin{cases} v_t^0 - d_2\Delta v^0 = 0 & \text{in } \Omega_T, \\ \frac{\partial v^0}{\partial \eta} = 0 & \text{on } \partial\Omega\times[0,T), \\ v^0(x,0) = v_0(x) & \text{on } \Omega. \end{cases} \quad (8)$$

Further, let $(u^k, v^k) \in L^2\big(0,T;H^2(\Omega)\big) \times L^2\big(0,T;H^2(\Omega)\big)$ for $k \in \mathbb{N}$ be a weak solution (9) and (10), with $u_t^{k+1}, v_t^{k+1} \in L^2\big(0,T;L^2(\Omega)\big)$, $\widehat{f}(u^k, v^k) = a_1 u^k$ and $\widehat{g}(u^k, v^k) = c_2 u^k v^k$:

$$\begin{cases} u_t^{k+1} - d_1\Delta u^{k+1} + (b_1 u^k + c_1 v^k)u^{k+1} = \widehat{f}(u^k, v^k) & \text{in } \Omega_T, \\ \frac{\partial u^{k+1}}{\partial \eta} = 0 & \text{on } \partial\Omega\times[0,T), \\ u^{k+1}(x,0) = u_0(x) & \text{on } \Omega; \end{cases} \quad (9)$$

$$\begin{cases} v_t^{k+1} - d_2 \Delta v^{k+1} + a_2 v^{k+1} = \widehat{g}(u^k, v^k) & \text{in } \Omega_T, \\ \frac{\partial v^{k+1}}{\partial \eta} = 0 & \text{on } \partial\Omega \times [0, T), \\ v^{k+1}(x, 0) = v_0(x) & \text{on } \Omega. \end{cases} \tag{10}$$

The following lemmas guarantee the existence and uniqueness of the functions (u^k, v^k) for $k \in \mathbb{N}_0$ in the sequence.

Lemma 2.4. *The system* (7)–(8) *possesses unique weak solutions, such that*

$$\begin{aligned} & u^0, v^0 \in L^\infty\big(0, T; H^1(\Omega)\big) \cap L^2\big(0, T; H^2(\Omega)\big), \\ & u_t^0, v_t^0 \in L^2\big(0, T; L^2(\Omega)\big), \end{aligned} \tag{11}$$

and standard regularity estimates hold true,

$$\operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} ||u^0(t)||_{H^1(\Omega)} + ||u^0||_{L^2(0,T;H^2(\Omega))} + ||u_t^0||_{L^2(0,T;L^2(\Omega))} \leqslant C(\Omega, T)||u_0||_{H^1(\Omega)}, \tag{12}$$

$$\operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} ||v^0(t)||_{H^1(\Omega)} + ||v^0||_{L^2(0,T;H^2(\Omega))} + ||v_t^0||_{L^2(0,T;L^2(\Omega))} \leqslant C(\Omega, T)||v_0||_{H^1(\Omega)}. \tag{13}$$

Proof. The existence and uniqueness of the weak solution (u^0, v^0) for system (7)–(8), which satisfies (11)–(13) follows from standard theory of parabolic equations (see Appendix Theorem 2.3). □

Lemma 2.5. *Under the assumptions of Theorem 2.2, the weak solution* (u^0, v^0) *of system* (7)–(8) *satisfies*

$$0 \leqslant u^0(x, t) \leqslant \gamma \text{ and } 0 \leqslant v^0(x, t) \leqslant \rho. \tag{14}$$

Proof. To prove that $u^0(x, t) \geqslant 0$. Let us take $\phi = (u^0)^- := \min\{0, u^0\}$ in the variational formulation of (7), we obtain

$$\begin{aligned} & \int_\Omega u_t^0 (u^0)^- \, dx + \int_\Omega d_1 \nabla u^0 \nabla (u^0)^- \, dx = 0, \\ & \frac{1}{2} \frac{d}{dt} \int_\Omega |(u^0)^-|^2 \, dx + d_1 \int_\Omega |\nabla (u^0)^-|^2 \, dx = 0. \end{aligned}$$

Integrating this over time, we obtain

$$\frac{1}{2} \int_\Omega |(u^0)^-(t)|^2 \, dx + d_1 \int_0^t \int_\Omega |\nabla (u^0)^-|^2 \, dx \, ds = \frac{1}{2} \int_\Omega |(u^0)^-(0)|^2 \, dx.$$

As $(u^0)^-(0) = (u_0)^- = 0$ we deduce

$$\frac{1}{2} \int_\Omega |(u^0)^-(t)|^2 \, dx + d_1 \int_0^t \int_\Omega |\nabla (u^0)^-|^2 \, dx \, ds = 0,$$

that is to say that $(u^0)^- = 0$ almost everywhere in Ω_T.

Now we establish a upper bound for u^0. If here we take $\phi = (u^0 - \gamma)^+ := \max\{u^0 - \gamma, 0\}$ in the variational formulation of (7), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(u^0-\gamma)^+|^2\,dx + d_1\int_\Omega |\nabla(u^0-\gamma)^+|^2\,dx = 0.$$

Integrating this over time, we get

$$\frac{1}{2}\int_\Omega |(u^0-\gamma)^+(t)|^2\,dx + d_1\int_0^t\int_\Omega |\nabla(u^0-\gamma)^+|^2\,dx\,ds = \frac{1}{2}\int_\Omega |(u^0-\gamma)^+(0)|^2\,dx.$$

As $(u^0-\gamma)^+(0) = (u_0-\gamma)^+ = 0$ we deduce

$$\frac{1}{2}\int_\Omega |(u^0-\gamma)^+(t)|^2\,dx + d_1\int_0^t\int_\Omega |\nabla(u^0-\gamma)^+|^2\,dx\,ds = 0,$$

that is to say that $(u^0-\gamma)^+ = 0$ almost everywhere in Ω_T.
The proof of $0 \leqslant v^0(x,t) \leqslant \rho$ does not differ from the one for $u^0(x,t)$ and it is therefore omitted here. □

Proof by induction

Lemma 2.6. *(Properties of the Iterative Sequence). Under the assumptions of Theorem 2.2, for every $k \in \mathbb{N}_0$, there is a unique weak solution to the system* (9)–(10) *such that*

$$\begin{aligned} &u^k, v^k \in L^\infty\big(0,T;H^1(\Omega)\big)\cap L^2\big(0,T;H^2(\Omega)\big),\\ &u_t^k, v_t^k \in L^2\big(0,T;L^2(\Omega)\big). \end{aligned} \tag{15}$$

Furthermore, for all $k\in\mathbb{N}_0$, the functions u^k, v^k satisfy the following inequalities:

$$0 \leqslant u^k(x,t) \leqslant \gamma,\ \ 0 \leqslant v^k(x,t) \leqslant \rho\ \ \textit{for a.e. } (x,t)\in\Omega_T, \tag{16}$$

and the standard regularity estimates

$$\begin{aligned} \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \|u^k(t)\|_{H^1(\Omega)} + \|u^k\|_{L^2(0,T;H^2(\Omega))} + \|u_t^k\|_{L^2(0,T;L^2(\Omega))}& \\ \leqslant 2C(\Omega,T)\|u_0\|_{H^1(\Omega)},& \end{aligned} \tag{17}$$

$$\begin{aligned} \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \|v^k(t)\|_{H^1(\Omega)} + \|v^k\|_{L^2(0,T;H^2(\Omega))} + \|v_t^k\|_{L^2(0,T;L^2(\Omega))}& \\ \leqslant C(\Omega,T)\big(\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^1(\Omega)}\big).& \end{aligned} \tag{18}$$

Proof. (i) First step: $k=0$. We first show that $\widehat{f}(u^0,v^0)\in L^2(0,T;L^2(\Omega))$:

$$\begin{aligned}\int_0^T \|\widehat{f}(u^0,v^0)\|^2_{L^2(\Omega)}\,dt &= \int_0^T \|a_1u^0\|^2_{L^2(\Omega)}\,dt\\ &= a_1^2\int_0^T \|u^0\|^2_{L^2(\Omega)}\,dt\\ &\leqslant a_1^2\int_0^T \|u^0\|^2_{H^1(\Omega)}\,dx\,dt\\ &\overset{(12)}{\leqslant} a_1^2\,C^2(\Omega,T)\,T_1\,\|u_0\|^2_{H^1(\Omega)}\\ &\leqslant \|u_0\|^2_{H^1(\Omega)}<\infty,\end{aligned}$$

where T_1 is such that $a_1^2\,C^2(\Omega,T)\,T_1\leqslant 1$.

Now let us prove that $\widehat{g}(u^0,v^0)\in L^2(0,T;L^2(\Omega))$:

$$\begin{aligned}\int_0^T \|\widehat{g}(u^0,v^0)\|^2{}_{L^2(\Omega)}\,dt &= \int_0^T \|c_2u^0v^0\|^2_{L^2(\Omega)}\,dt\\ &\leqslant\ c_2^2\int_0^T \|u^0v^0\|^2_{L^2(\Omega)}\,dt\\ &=\ c_2^2\int_0^T\int_\Omega |u^0|^2\,|v^0|^2\,dx\,dt\\ &\overset{\text{Young}}{\leqslant}\ \frac{1}{2}c_2^2\int_0^T\int_\Omega |u^0|^4+|v^0|^4\,dx\,dt\\ &\leqslant\ \frac{1}{2}c_2^2\int_0^T \|u^0\|^4_{L^4(\Omega)}+\|v^0\|^4_{L^4(\Omega)}\,dt\\ &\leqslant\ \frac{1}{2}c_2^2\,C^4\int_0^T \|u^0\|^4_{H^1(\Omega)}+\|v^0\|^4_{H^1(\Omega)}\,dt\\ &\overset{(12),\,(13)}{\leqslant}\ c_2^2\,C^4\,C^4(\Omega,T)\,T_2\left(\|u_0\|^4_{H^1(\Omega)}+\|v_0\|^4_{H^1(\Omega)}\right)<\infty,\end{aligned}$$

with C an adequate embedding constant and

$$T_2:=\min\left\{1,\frac{1}{c_2^2\,C^4\,C^4(\Omega,T)}\right\}.$$

Therefore, by Theorem 2.3, there is a unique weak solution (u^1,v^1) of the system (9)–(10) that satisfies (15), (17) and (18).

Now, we only have to establish (16) for u^1 and v^1. To show that $u^1\geqslant 0$, we take as a test function $(u^1)^-:=\min\{0,u^1\}$ in the weak formulation of (9),

$$\begin{aligned}\int_\Omega u_t^1(u^1)^-\,dx+d_1\int_\Omega \nabla u^1\nabla(u^1)^-\,dx &= a_1\int_\Omega u^0(u^1)^-\,dx\\ &\quad-\int_\Omega (b_1u^0+c_1v^0)u^1(u^1)^-\,dx.\end{aligned}$$

In view the right-hand side is negative, then

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|(u^1)^-|^2\,dx + d_1\int_{\Omega}|\nabla(u^1)^-|^2\,dx \leqslant 0.$$

Integrating this over time, we obtain

$$\frac{1}{2}\int_{\Omega}|(u^1)^-(t)|^2\,dx + d_1\int_0^t\int_{\Omega}|\nabla(u^1)^-|^2\,dx\,ds \leqslant \frac{1}{2}\int_{\Omega}|(u^1)^-(0)|^2\,dx.$$

As $(u^1)^-(0) = u_0^- = 0$ we deduce

$$\frac{1}{2}\int_{\Omega}|(u^1)^-(t)|^2\,dx + d_1\int_0^t\int_{\Omega}|\nabla(u^1)^-|^2\,dx\,ds \leqslant 0,$$

that is to say that $(u^1)^- = 0$ almost everywhere in Ω_T.

To show that $u^1(x,t) \leqslant \gamma$ almost everywhere in Ω_T, we take as a test function $(u^1-\gamma)^+ := \max\{0, u^1-\gamma\}$ in the weak formulation of (9), this yields

$$\begin{aligned}
\int_{\Omega} u_t^1(u^1-\gamma)^+\,dx &+ d_1\int_{\Omega}\nabla u^1\nabla(u^1-\gamma)^+\,dx \\
&= \int_{\Omega}(a_1u^0 - b_1u^0u^1 - c_1v^0u^1)(u^1-\gamma)^+\,dx \\
&= \int_{\Omega}(a_1u^0 - b_1u^0\gamma + b_1u^0\gamma - b_1u^0u^1 - c_1v^0u^1)(u^1-\gamma)^+\,dx \\
&= \int_{\Omega}\big((a_1-b_1\gamma)u^0 - b_1u^0(u^1-\gamma) - c_1v^0u^1\big)(u^1-\gamma)^+dx \\
&= -\int_{\Omega} b_1u^0(u^1-\gamma)(u^1-\gamma)^+ + c_1v^0u^1(u^1-\gamma)^+\,dx \\
&= -\int_{\Omega} b_1u^0|(u^1-\gamma)^+|^2 + c_1v^0u^1(u^1-\gamma)^+\,dx.
\end{aligned}$$

Hence

$$\frac{1}{2}\frac{d}{dt}||(u^1-\gamma)^+||^2_{L^2(\Omega)} + d_1\int_{\Omega}||\nabla(u^1-\gamma)^+||^2\,dx \leqslant 0$$

and

$$||(u^1-\gamma)^+||^2_{L^2(\Omega)} + d_1\int_0^t\int_{\Omega}||\nabla(u^1-\gamma)^+||^2\,dx \leqslant ||(u^1-\gamma)^+(0)||^2_{L^2(\Omega)},$$

since $(u^1-\gamma)^+(0) = (u_0-\gamma)^+ = 0$, follows $u^1(x,t) \leqslant \gamma$ almost everywhere in Ω_T.

Now, we prove that $v^1(x,t) \geqslant 0$ almost everywhere in Ω_T. Multiplying (10) by $(v^1)^- = \min\{0, v^1\}$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|(v^1)^-|^2\,dx + d_2\int_{\Omega}|\nabla(v^1)^-|^2\,dx + a_2\int_{\Omega}|(v^1)^-|^2dx = \int_{\Omega}c_2u^0v^0(v^1)^-\,dx,$$

then it follows that

$$\frac{d}{dt}\int_{\Omega}|(v^1)^-|^2\,dx \leqslant 0,$$

and thus

$$\int_{\Omega}|(v^1)^-|^2\,dx \leqslant \int_{\Omega}|(v^1)^-(0)|^2\,dx = 0.$$

This implies that $v^1(x,t) \geqslant 0$ almost everywhere in Ω_T.

Let us show that ρ is an upper bound of v^1. Testing the weak formulation of the system (10) using $(v^1-\rho)^+$, one gets

$$\begin{aligned}
\int_{\Omega} v_t^1(v^1-\rho)^+\,dx &+ d_2\int_{\Omega}\nabla v^1\nabla(v^1-\rho)^+\,dx\\
&= \int_{\Omega}(c_2u^0v^0 - a_2v^1)(v^1-\rho)^+\,dx\\
&= \int_{\Omega}(c_2u^0v^0 + c_2\gamma v^0 - c_2\gamma v^0 - a_2v^1 + a_2\rho - a_2\rho)(v^1-\rho)^+\,dx\\
&= \int_{\Omega}\big(c_2(u^0-\gamma)v^0 + (c_2\gamma v^0 - a_2\rho) - a_2(v^1-\rho)\big)(v^1-\rho)^+\,dx\\
&= \int_{\Omega} c_2(u^0-\gamma)v^0(v^1-\rho)^+ + (c_2\gamma v^0 - a_2\rho)(v^1-\rho)^+\\
&\qquad\qquad - a_2|(v^1-\rho)^+|^2\,dx,
\end{aligned}$$

since $u^0 \leqslant \gamma \leqslant a_2/c_2$ and $0 \leqslant v^0 \leqslant \rho$. Then

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|(v^1-\rho)^+|^2\,dx + d_2\int_{\Omega}|\nabla(v^1-\rho)^+|^2\,dx \leqslant 0.$$

Integrating on $(0,t)$ gives

$$\frac{1}{2}\int_{\Omega}|(v^1-\rho)^+|^2\,dx + d_2\int_0^t\int_{\Omega}|\nabla(v^1-\rho)^+|^2\,dx \leqslant |(v^1-\rho)^+(0)|^2 = 0.$$

Thus, we conclude that $v^1(x,t) \leqslant \rho$ almost everywhere in Ω_T.

(ii) Induction hypothesis: the lemma holds for an arbitrary $k \in \mathbb{N}_0$.
(iii) Induction step: by the induction hypothesis we have, $u^k, v^k \in L^\infty(\Omega_T)$ and $\widehat{f}, \widehat{g} \in L^2(0,T;\Omega)$. Then Theorem 2.3 guarantees the existence and uniqueness of u^{k+1} and v^{k+1}. The proof of $0 \leqslant u^{k+1} \leqslant \gamma$ and $0 \leqslant v^{k+1} \leqslant \rho$ follows the same steps used for the case $k=0$ and it is therefore omitted here. Hence, Lemma 2.6 is satisfied for $(k+1)$ and this concludes the induction proof. □

Results

Now we prove the main results of this paper. The existence and uniqueness of weak solution for system (1) (Theorem 2.2) as well as its global existence in time (Theorem 5.5).

Existence and uniqueness of weak solutions

Before the proof of Theorem 2.2, we subtract the $(k+1)$-th and k-th terms of the iterative sequence (9) and (10) to obtain

$$(u^{k+1}-u^k)_t - d_1\Delta(u^{k+1}-u^k) + b_1u^ku^{k+1} - b_1u^{k-1}u^k \\ + c_1v^ku^{k+1} - c_1v^{k-1}u^k = a_1u^k - a_1u^{k-1} \quad (19)$$

and

$$(v^{k+1}-v^k)_t - d_2\Delta(v^{k+1}-v^k) + a_2v^{k+1} - a_2v^k = c_2u^kv^k - c_2u^{k-1}v^{k-1}. \quad (20)$$

After subtracting and adding $b_1u^ku^k + c_1v^ku^k$ in (19) and doing the same with $c_2u^kv^{k-1}$ in (20) we obtain

$$(u^{k+1}-u^k)_t - d_1\Delta(u^{k+1}-u^k) + b_1u^k(u^{k+1}-u^k) + c_1v^k(u^{k+1}-u^k) \\ = a_1(u^k-u^{k-1}) - b_1u^k(u^k-u^{k-1}) - c_1u^k(v^k-v^{k-1}) \quad (21)$$

and

$$(v^{k+1}-v^k)_t - d_2\Delta(v^{k+1}-v^k) + a_2(v^{k+1}-v^k) \\ = c_2u^k(v^k-v^{k-1}) + c_2v^{k-1}(u^k-u^{k-1}). \quad (22)$$

Now, let us denote by

$$w^{k+1} := u^{k+1}-u^k, \qquad \tilde{f}^k := (a_1-b_1u^k)w^k - c_1u^kz^k,$$
$$z^{k+1} := v^{k+1}-v^k, \qquad \tilde{g}^k := c_2u^kz^k + c_2v^{k-1}w^k,$$

for $k = 0, 1, \ldots$. Then, from equations (21) and (22) we get the following boundary value problems

$$\begin{cases} w_t^{k+1} - d_1\Delta w^{k+1} + b_1u^kw^{k+1} + c_1v^kw^{k+1} = \tilde{f}^k & \text{in } \Omega_T, \\ \frac{\partial w^{k+1}}{\partial \eta} = 0 & \text{on } \partial\Omega\times[0,T), \\ w^{k+1}(x,0) = 0 & \text{on } \Omega; \end{cases} \quad (23)$$

and

$$\begin{cases} z_t^{k+1} - d_2\Delta z^{k+1} + a_2 z^{k+1} = \tilde{g}^k & \text{in } \Omega_T, \\ \frac{\partial z^{k+1}}{\partial \eta} = 0 & \text{on } \partial\Omega \times [0,T), \\ z^{k+1}(x,0) = 0 & \text{on } \Omega. \end{cases} \tag{24}$$

Proof of Theorem 2.2. To prove existence, we show that the iterative sequences $\{u^k\}, \{v^k\}$ obtained are Cauchy sequences. Since, $H^1(\Omega)$ is a Banach space, there are u and v limiting functions that fulfill the system (1).

Let $k \in \mathbb{N}$. From (23) and inequality (6), follows

$$\begin{aligned} ||w^{k+1}||^2_{L^\infty(0,T;H^1(\Omega))} &\leqslant C(\Omega,T)\int_0^T ||\tilde{f}^k||^2_{L^2(\Omega)}\,dt \\ &\leqslant C(\Omega,T)\int_0^T\int_\Omega |(a_1 - b_1u^k)w^k - c_1u^kz^k|^2\,dx\,dt \\ &\leqslant C(\Omega,T)\int_0^T\int_\Omega 2|(a_1 - b_1u^k)w^k|^2 + 2|c_1u^kz^k|^2\,dx\,dt \\ &\leqslant C(\Omega,T)\int_0^T\int_\Omega 4(|a_1|^2 + |b_1u^k|^2)|w^k|^2 + 2|c_1u^kz^k|^2\,dx\,dt \\ &\leqslant C(\Omega,T)\left(4(a_1^2 + b_1^2\gamma^2)\int_0^T\int_\Omega |w^k|^2\,dx\,dt + 2c_1^2\gamma^2\int_0^T\int_\Omega |z^k|^2\,dx\,dt\right) \\ &\leqslant C(\Omega,T)\left(8a_1^2\int_0^T ||w^k||^2_{L^2(\Omega)}\,dt + 2c_1^2\gamma^2\int_0^T ||z^k||^2_{L^2(\Omega)}\,dt\right) \\ &\leqslant C(\Omega,T)\left(8a_1^2\int_0^T ||w^k||^2_{H^1(\Omega)}\,dt + 2c_1^2\gamma^2\int_0^T ||z^k||^2_{H^1(\Omega)}\,dt\right) \\ &\leqslant 8a_1^2C(\Omega,T)T||w^k||^2_{L^\infty(0,T;H^1(\Omega))} + 2c_1^2\gamma^2C(\Omega,T)T||z^k||^2_{L^\infty(0,T;H^1(\Omega))}. \end{aligned} \tag{25}$$

If $0 \leqslant T \leqslant T_3$, with $T_3 := \min\left\{\frac{1}{4}, \frac{1}{32a_1^2C(\Omega,T)}, \frac{1}{8c_1^2\gamma^2C(\Omega,T)}\right\}$, then

$$||w^{k+1}||^2_{L^\infty(0,T;H^1(\Omega))} \leqslant \frac{1}{4}(||w^k||^2_{L^\infty(0,T;H^1(\Omega))} + ||z^k||^2_{L^\infty(0,T;H^1(\Omega))}). \tag{26}$$

Similarly, from (24) and inequality (6), we get

$$\begin{aligned} ||z^{k+1}||^2_{L^\infty(0,T;H^1(\Omega))} &\leqslant C(\Omega,T)\int_0^T ||g^k||^2_{L^2(\Omega)}\,dt \\ &= C(\Omega,T)\int_0^T ||c_2u^kz^k + c_2v^{k-1}w^k||^2_{L^2(\Omega)}\,dt \\ &\leqslant c_2^2\,C(\Omega,T)\int_0^T 2||u^kz^k||^2_{L^2(\Omega)} + 2||v^{k-1}w^k||^2_{L^2(\Omega)}\,dt \end{aligned} \tag{27}$$

$$\leqslant 2c_2^2 C(\Omega,T)\Big(||u^k||^2_{L^\infty(\Omega)}\int_0^T ||z^k||^2_{L^2(\Omega)}dt + ||v^{k-1}||^2_{L^\infty(\Omega)}\int_0^T ||w^k||^2_{L^2(\Omega)}dt\Big)$$
$$\leqslant 2c_2^2 C(\Omega,T)\Big(\gamma^2\int_0^T ||z^k||^2_{H^1(\Omega)}dt + \rho^2\int_0^T ||w^k||^2_{H^1(\Omega)}dt\Big)$$
$$\leqslant 2c_2^2 C(\Omega,T)\Big(\gamma^2 T_4||z^k||^2_{L^\infty(0,T;H^1(\Omega))} + \rho^2 T_4||w^k||^2_{L^\infty(0,T;H^1(\Omega))}\Big), \qquad (28)$$

where

$$T_4 := \min\left\{\frac{1}{4}, \frac{1}{8c_2^2\gamma^2 C(\Omega,T)}, \frac{1}{8c_2^2\rho^2 C(\Omega,T)}\right\}.$$

Then the inequality (28) becomes

$$||z^{k+1}||^2_{L^\infty(0,T;H^1(\Omega))} \leqslant \frac{1}{4}\Big(||w^k)||^2_{L^\infty(0,T;H^1(\Omega))} + ||z^k||^2_{L^\infty(0,T;H^1(\Omega))}\Big). \qquad (29)$$

Finally, from (26) and (29)

$$||u^{k+1}-u^k||^2_{L^\infty(0,T;H^1(\Omega))} + ||v^{k+1}-v^k||^2_{L^\infty(0,T;H^1(\Omega))}$$
$$\leqslant \frac{1}{2}\Big(||(u^k-u^{k-1})||^2_{L^\infty(0,T;H^1(\Omega))} + ||v^k-v^{k-1}||^2_{L^\infty(0,T;H^1(\Omega))}\Big). \qquad (30)$$

Thus, $\{u^k\}$ and $\{v^k\}$ are Cauchy sequences in $L^\infty(0,T;H^1(\Omega))$ and there are functions u and v in $L^\infty(0,T;H^1(\Omega))$ such that

$$u^k \to u \ \text{ and } \ v^k \to v \ \text{ strongly in } L^\infty(0,T;H^1(\Omega)).$$

As $L^2(0,T;H^2(\Omega))$ and $L^2(0,T;L^2(\Omega))$ are Hilbert spaces, the uniform bounds (17) and (18) imply that for subsequences u^{k_l} and v^{k_l}

$$\begin{aligned} &u^{k_l} \rightharpoonup u; \quad v^{k_l} \rightharpoonup v \quad \text{weakly in } L^2(0;T;H^2(\Omega));\\ &u_t^{k_l} \rightharpoonup u_t; \quad v_t^{k_l} \rightharpoonup v_t \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \end{aligned} \qquad (31)$$

Now we take the limit $l \to \infty$ and by (31), we obtain a weak solution (u,v) which satisfies system (1).

In order to ensure the uniqueness, we suppose that (u_1,v_1) and (u_2,v_2) are the weak solutions of system (1). Owing to (30), it follows that:

$$||u_1-u_2||^2_{L^\infty(0,T;H^1(\Omega))} + ||v_1-v_2||^2_{L^\infty(0,T;H^1(\Omega))}$$
$$\leqslant \frac{1}{2}\Big(||u_1-u_2||^2_{L^\infty(0,T;H^1(\Omega))} + ||v_1-v_2||^2_{L^\infty(0,T;H^1(\Omega))}\Big).$$

Therefore, $(u_1,v_1)=(u_2,v_2)$ in $L^\infty(0,T;H^1(\Omega))\times L^\infty(0,T;H^1(\Omega))$, i.e. the weak solution of system (1) is unique. □

Global existence of solutions

In order to prove global existence in time of solutions for system (1), we apply the technique based on the bounded invariant regions [11]. With this purpose, we write system (1) with predator-prey interaction functions (2) and (3) in matrix form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + \begin{pmatrix} u(a_1 - b_1 u) - c_1 uv \\ c_2 uv - a_2 v \end{pmatrix}. \tag{32}$$

As in [11], we define an invariant region for (32).

Definition 5.1. Let $\Sigma \subset \mathbb{R}^m$ be a closed set. Σ is a (positively) invariant region for the local solution of (32), if any solution $(u(x,t), v(x,t))$ having all of its boundary and initial values in Σ, satisfies $(u(x,t), v(x,t)) \in \Sigma$ for all $x \in \Omega$ and for all $t \in [0, \delta)$.

To find invariant regions, we use the following combined version of the well known results (see [11], Chapter 14, §B).

Theorem 5.2. *Suppose that D is the diagonal matrix with entries d_1 and d_2, then any region of the form*

$$\Sigma = \{u : \theta_1 \leqslant u \leqslant \eta_1\} \cap \{v : \theta_2 \leqslant v \leqslant \eta_2\},$$

is invariant for (32), *whenever the following conditions are satisfied*

$$\nabla(\theta_1 - u) \cdot V|_{u=\theta_1} \leqslant 0 \ \ in\ \Sigma, \tag{33}$$

$$\nabla(u - \eta_1) \cdot V|_{u=\eta_1} \leqslant 0 \ \ in\ \Sigma, \tag{34}$$

$$\nabla(\theta_2 - v) \cdot V|_{v=\theta_2} \leqslant 0 \ \ in\ \Sigma, \tag{35}$$

$$\nabla(v - \eta_2) \cdot V|_{v=\eta_2} \leqslant 0 \ \ in\ \Sigma, \tag{36}$$

where $V = [u(a_1 - b_1 u) - c_1 uv,\ c_2 uv - a_2 v]$.

Lemma 5.3. *Let* $\eta = \dfrac{a_1}{b_1}$, $\dfrac{a_1}{b_1} \leqslant \dfrac{a_2}{c_2}$, $0 \leqslant \rho$, *and*

$$\Sigma = \{(u,v) : 0 \leqslant u \leqslant \eta, 0 \leqslant v \leqslant \rho\}.$$

Then Σ is an invariant region for the system (32).

Proof. To prove that Σ is an invariant region for system (32), we verify that inequalities (33)–(36) hold

- If $G(u,v) = -u$,

$$\nabla G \cdot V|_{u=0} = -(u(a_1 - b_1 u) - c_1 uv)|_{u=0} = 0 \ \text{ in } \Sigma, \text{ so } u \geqslant 0.$$

- If $G(u,v) = -v$,

$$\nabla G \cdot V|_{v=0} = -(c_2uv - a_2v)|_{v=0} = 0 \text{ in } \Sigma, \text{ so } v \geqslant 0.$$

- If $G(u,v) = u - \eta$ and since $\eta = a_1/b_1$, we obtain

$$\begin{aligned}\nabla G \cdot V|_{u=\eta} &= (u(a_1 - b_1u) - c_1uv)|_{u=\eta}\\ &= \eta(a_1 - b_1\eta - c_1v)\\ &= -\frac{a_1}{b_1}c_1v \leqslant 0 \text{ in } \Sigma, \text{ so } u \leqslant \eta.\end{aligned}$$

- $G(u,v) = v - \rho$ and since $\frac{a_1}{b_1} \leqslant \frac{a_2}{c_2}$, we obtain

$$\begin{aligned}\nabla G \cdot V|_{v=\rho} &= (c_2uv - a_2v)|_{v=\rho}\\ &= \rho(c_2u - a_2)\\ &\leqslant \rho(c_2\frac{a_1}{b_1} - a_2) \leqslant 0 \text{ in } \Sigma, \text{ so } v \leqslant \rho.\end{aligned}$$

Therefore Σ is an invariant set by Theorem 5.2. □

Remark 5.4. Note that system (32) admits arbitrarily large bounded invariant rectangles in the positive strip $0 \leqslant u \leqslant \frac{a_1}{b_1}$, $v \geqslant 0$.

Theorem 5.5. *If the initial conditions u_0, v_0 are bounded, uniformly continuous functions on $\mathbb{R}^n$, such that* $\lim_{|x|\to\infty} |u_0(x)| = 0$, $\lim_{|x|\to\infty} |v_0(x)| = 0$ *and* $(u_0(x), v_0(x)) \in \Sigma$ *for every $x \in \Omega$, and in addition*

$$\frac{a_1}{b_1} \leqslant \frac{a_2}{c_2}.$$

Then the solution (u,v) of the system (32) *exists globally in time.*

Proof. Applying Corollary 14.9 from [11], the result follows. □

Numerical simulations

In this section, in order to illustrate the convergence of the iterative method introduced in this paper, we present two numerical examples. To generate the sequence of solutions (u^k, v^k) of (9) and (10), we use the backward Euler method and the finite element method, for time direction and spatial direction, respectively.

Let us consider the finite-dimensional space of piecewise linear functions $V_h \subset H^1(\Omega)$ with basis functions $\{\phi_i\}$, $i = 1,2,\dots,N$. Then, we look for functions $(u^{k+1}(t), v^{k+1}(t)) \in L^2(0,T;V_h) \times L^2(0,T;V_h)$ such that

$$\begin{aligned} M u_t^{k+1} + d_1 A u^{k+1} + M D(u^k, v^k) u^{k+1} &= F^k(t), \\ M v_t^{k+1} + d_2 A v^{k+1} + a_2 M v^{k+1} &= G^k(t). \end{aligned} \tag{37}$$

Where the matrices M, A, $D(u^k, v^k) \in \mathbb{R}^N \times \mathbb{R}^N$ are given by

$$M_{ij} = \int_\Omega \phi_j \phi_i dx, \quad A_{ij} = \int_\Omega \nabla\phi_j \nabla\phi_i dx, \quad \text{and}$$
$$D(u^k, v^k) = \operatorname{diag}(b_1 u_1^k + c_1 v_1^k,\, b_1 u_2^k + c_1 v_2^k, \dots,\, b_1 u_N^k + c_1 v_N^k),$$

and the right-hand sides $F^k(t)$, $G^k(t) \in \mathbb{R}^N$ are given by

$$F^k(t) = \int_\Omega \widehat{f}(u^k, v^k)\phi_i dx, \text{ and } G^k(t) = \int_\Omega \widehat{g}(u^k, v^k)\phi_i dx.$$

For the time discretization we split the time interval into sub-intervals with constant step $\Delta t > 0$, the backward Euler method for (37) reads

$$\begin{aligned} (M + d_1 \Delta t\, A + \Delta t\, M D(u^k, v^k))\, u^{k+1}(t+\Delta t) &= M u^{k+1}(t) + \Delta t\, F^k(t), \\ (M + d_2 \Delta t\, A + \Delta t\, M)\, v^{k+1}(t+\Delta t) &= M v^{k+1}(t) + \Delta t\, G^k(t). \end{aligned} \tag{38}$$

Solving the linear system (38) for $k = 0,1,2,\dots$, allows us to generate the sequence (u^{k+1}, v^{k+1}). In this way we would conclude that the series of iterates converges to the solution of problem (1) if the values of the expression

$$||u^{k+1} - u^k||_{L^2(0;T;H^1(\Omega))} + ||v^{k+1} - v^k||_{L^2(0;T;H^1(\Omega))}$$

decrease towards 0 sufficiently fast as the iteration counter k increases.

Example 6.1: To check the theoretically predicted convergence of the proposed method, we study a practical example of the Lotka–Volterra diffusion system with logistic growth of preys (1) with $\Omega = [0,1]$. The calculations were based on the above finite element scheme, we use grid with equal size step $h = 0.032$, time step $\Delta t = 0.0032$, and take the following ecological parameters: $d_1 = 0.05$ $d_2 = 25$, $a_1 = 4$, $a_2 = 4$, $b_1 = 2$, $c_1 = 4$, and $c_2 = 1$ such that the hypothesis of Theorem 2.2 holds. The initial distribution of preys and predators is given by the following functions:

$$u_0(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant \frac{1}{8}, \\ 0 & \text{if } \frac{1}{8} < x < \frac{7}{8}, \\ 1 & \text{if } \frac{7}{8} \leqslant x \leqslant 1; \end{cases} \qquad v_0(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \frac{3}{8}, \\ 5 & \text{if } \frac{3}{8} < x < \frac{5}{8}, \\ 0 & \text{if } \frac{5}{8} \leqslant x \leqslant 1. \end{cases}$$

Fig. 1 shows the convergence history of the generated sequence. The tolerance of the $L^2(0,T;H^1(\Omega))$-norm changes between two successive iterates, which monitors the convergence of the iterative method, has been set equal to 10^{-8}. The method converges in 17 iterations. In Fig. 2 and 3, we observe the convergence of the iterative method towards the solution of the prey-predator system (1). The iterate (u^k, v^k) for $k = 1, 2, \ldots$, are denoted by the k in the plots.

From the qualitative point of view of the solutions, Fig. 2 and 3 show clearly the effect of the diffusive spatial dispersion of the two populations. At first, both preys and predators move towards the regions of low density until they are homogeneously distributed in the interval $[0, 1]$. After that, the population of preys grows logistically until they reach the steady-state, i.e the carrying capacity $\gamma = 4/2$. In contrast, even though the interaction with preys and its abundance, the predator population decay continuously until extinction (steady-state for predators).

Example 6.2: We next examine the convergence behavior of the iterative method with the following parameters $d_1 = 0.05$, $d_2 = 5$, $a_1 = 4$, $a_2 = 4$, $b_1 = 1$, $c_1 = 4$ and $c_2 = 16$. The size of the spatial grid, the time step and the intial conditions $u_0(x)$, $v_0(x)$ are chosen as in the previous example.

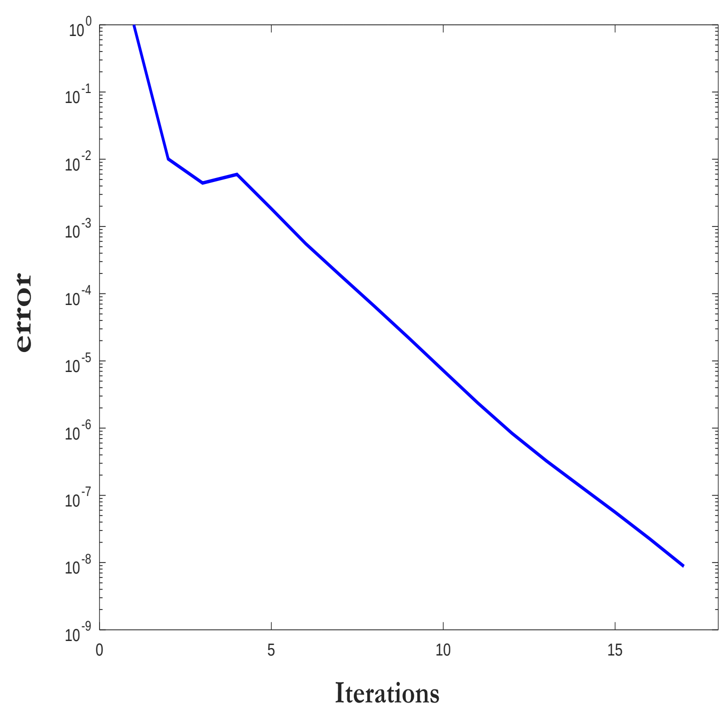

Figure 1. Behavior of the convegence error ϵ as a function of the number of iterations, where $\epsilon = ||u^{k+1} - u^k||_{L^2(0;T;H^1(\Omega))} + ||v^{k+1} - v^k||_{L^2(0;T;H^1(\Omega))}$ here.

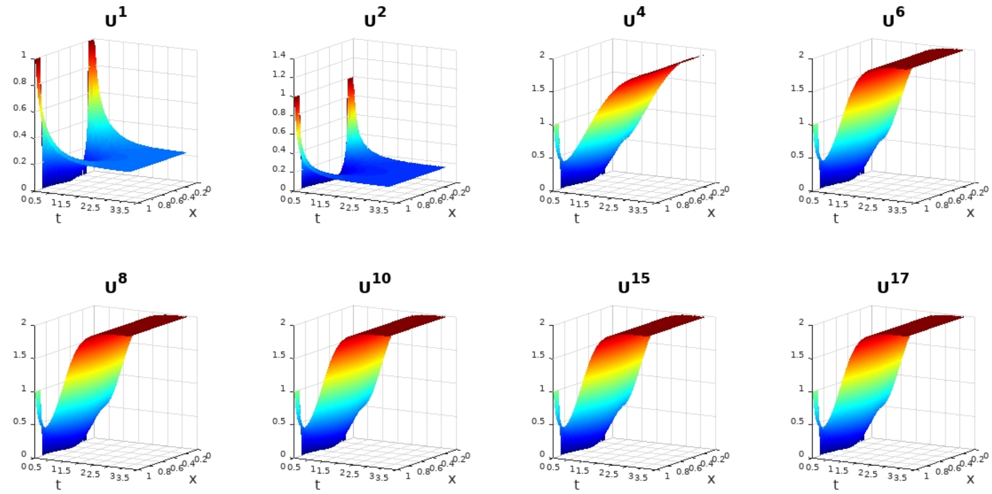

Figure 2. Convergence of the iterative method for preys. Iterations $k = 1, 2, 4, 6, 8, 10,$ $15, 17$.

We begin by providing the convergence history of the method in Fig. 4. The error has a fluctuating behavior and it takes 141 iteration to fulfill the stopping criteria of 10^{-8}. Note that, in this case the requirement that $a_1/b_1 \leqslant a_2/c_2$ in Theorem 2.2 is not fulfilled, however this is a sufficient and not necessary condition for convergence.

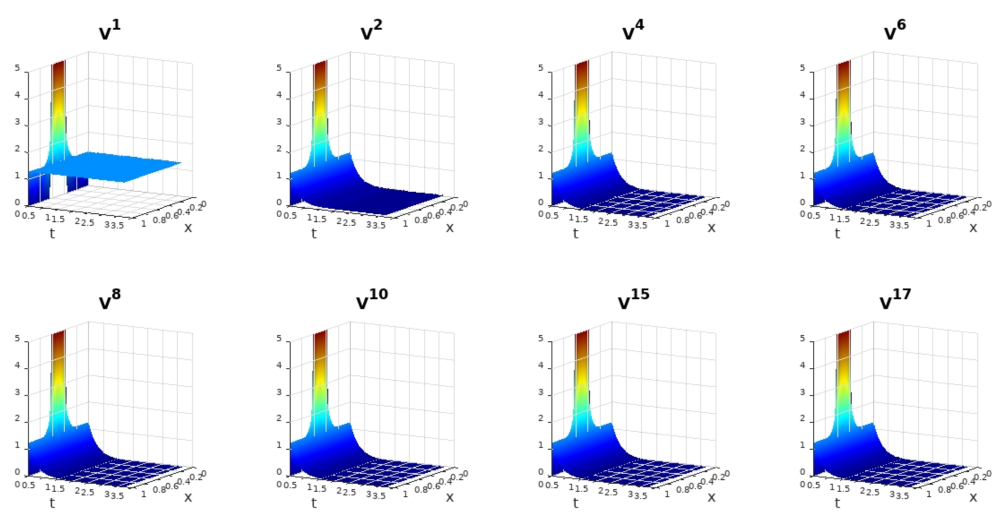

Figure 3. Convergence of the iterative method for predators. Iterations $k = 1, 2, 4, 6, 8,$ $10, 15, 17$.

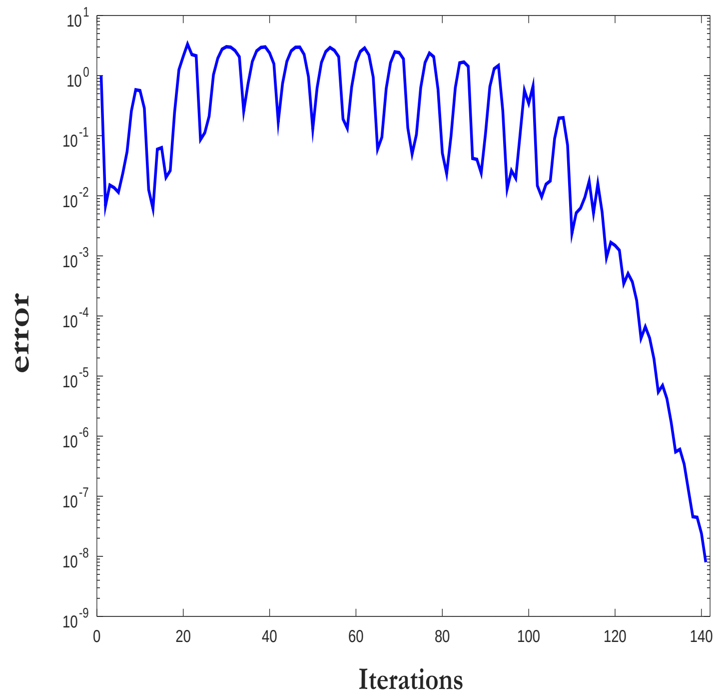

Figure 4. Behavior of the convegence error ϵ as a function of the number of iteration, where $\epsilon = ||u^{k+1} - u^k||_{L^2(0;T;H^1(\Omega))} + ||v^{k+1} - v^k||_{L^2(0;T;H^1(\Omega))}$.

Looking at the results $u^{141}(x,t)$ and $v^{141}(x,t)$ in the Fig. 5 and 6, we see that after both population fall rapidly as they move away from their original locations, the prey and predator populations oscillate over time until they reach a coexistence steady-state. The inequality $a_1/b_1 \leqslant a_2/c_2$ states that for there to be a steady state with prey and predator both present, the carrying capacity of the prey, a_1/b_1, must be high enough to support the predator.

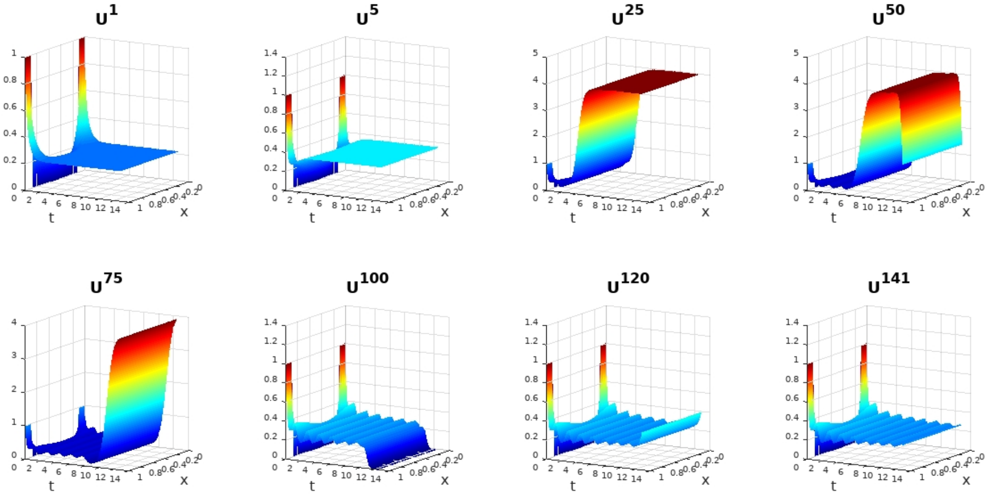

Figure 5. Convergence of the iterative method for preys. Iterations $k = 1$, 5, 10, 25, 50, 75, 100, 120, 141.

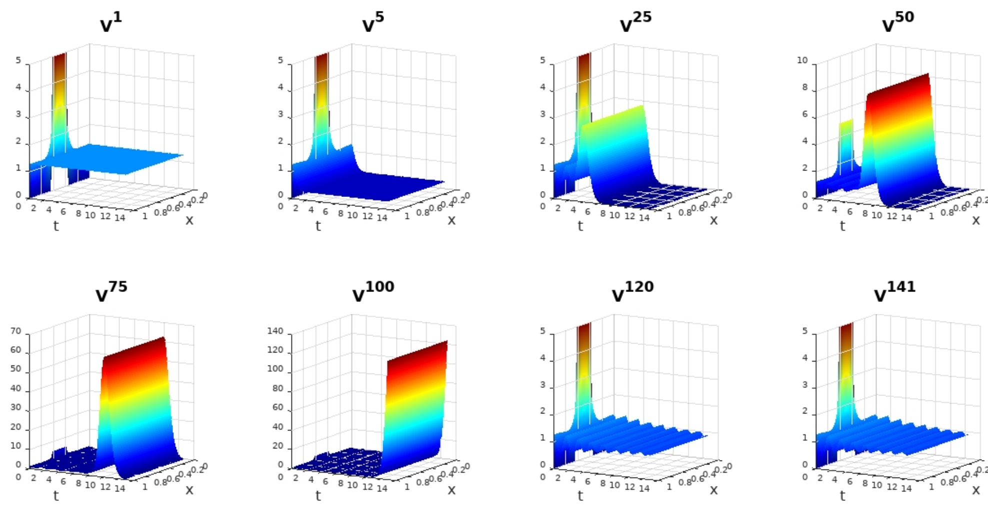

Figure 6. Convergence of the iterative method for predators. Iterations $k = 1, 5, 10, 25, 50, 75, 100, 120, 141$.

Conclusions

Using an alternative method based on successive iterations of a linear approximation of the original problem and under mild regularity conditions, we proved the existence, uniqueness and positivity of the solution for the predator-prey model with logistic growth of preys. The proof is based on mathematical induction and takes full advantage of the standard theory of linear partial differential equations. In order to guarantee convergence of the iterative method (i.e. the existence of local solution) and the global existence of the solution; it is sufficient that, the carrying capacity of the environment being bounded by the ratio between the death rate of the predators and growth rate of predators due to the interaction with preys. Numerical tests verify that the proposed iteration is convergent to the solution of the model and that the solution exists globally in time.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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Existencia de una solución local y global para un modelo depredador-presa espacio temporal

Resumen: En este artículo demostramos la existencia y la unicidad de soluciones débiles para una especie de sistema Lotka-Volterra, mediante el uso de técnicas de linealización sucesiva. Este enfoque tiene la ventaja de tratar dos ecuaciones por separado en cada paso iterativo. Bajo condiciones iniciales adecuadas, construimos una región invariante para mostrar la existencia global en el tiempo de soluciones para el sistema. Mediante encajamientos de Sobolev y resultados de regularidad, encontramos estimaciones para las poblaciones de depredadores y presas en normas apropiadas. Para demostrar las propiedades de convergencia del método introducido, se presentan varios ejemplos numéricos.

Palabras clave: solución débil global; método iterativo; sistema depredador-presa.

Existência de uma solução local e global para um modelo predador-presa espaço temporal

Resumo: Neste artigo provamos a existência e a unicidade de soluções fracas para um tipo de sistema Lotka-Volterra, usando técnicas de linearização sucessiva. Essa abordagem tem a vantagem de tratar duas equações separadamente em cada etapa da iteração. Sob condições iniciais adequadas, construímos uma região invariável para mostrar a existência global em tempo de soluções para o sistema. Por meio de imersões de Sobolev e resultados de regularidade, encontramos estimativas para populações de predadores e presas em normas apropiadas. Para demonstrar as propriedades de convergência do método introduzido, vários exemplos numéricos são apresentados.

Palavras-chave: solução fraca global; método iterativo; sistema predador-presa.

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