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ORIGINAL ARTICLE

A local Jacobian smoothing method for solving Nonlinear Complementarity Problems

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Abstract

In this paper, we present a smoothing of a family of nonlinear complementarity functions and use its properties in combination with the smooth Jacobian strategy to present a new generalized Newton-type algorithm to solve a nonsmooth system of equations equivalent to the Nonlinear Complementarity Problem. In addition, we prove that the algorithm converges locally and q-quadratically, and analyze its numerical performance.

Keywords: nonlinear complementarity problem; complementarity function; generalized Newton method; Q-quadratic convergence.

Introduction

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function. The Nonlinear Complementarity Problem, NCP for short, is to find a vector $x \in \mathbb{R}^n$ satisfying the following conditions

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0.$$

We say that a vector in $\mathbb{R}^n$ is nonnegative if each of its components is nonnegative.

There are numerous and varied applications of the NCP in Engineering [1, 2] and Economics [3, 4]. In this last area, complementarity and economic equilibrium are synonymous.

One technique, perhaps the most popular, to solve nonlinear complementarity problems is to write them in an equivalent way as a nonlinear system of equations. In this process, it is used a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(a,b)=0 \Longleftrightarrow a\geq 0,\ b\geq 0,\ ab=0, \tag{1}$$

called a *complementarity function* [5].

The equivalence (1) allows to conclude that a *complementarity function* is nondifferentiable due to the lack of smoothness of its trace by the intersection with the xy plane which is not differentiable at $(0,0)$.

Given a *complementarity function* φ and a function $\Phi:\mathbb{R}^n\to\mathbb{R}^n$, we define the nonlinear system of equations by

$$\Phi(\boldsymbol{x})=\begin{pmatrix}\varphi(x_1,F_1(\boldsymbol{x}))\\ \vdots \\ \varphi(x_n,F_n(\boldsymbol{x}))\end{pmatrix}=0, \tag{2}$$

which is nondifferentiable due to the lack of smoothness of φ. From (1), it follows that a necessary and sufficient condition for a vector $\boldsymbol{x}_*$ to solve the NCP is that this vector solves the system (2).

Two examples of complementarity functions, widely used, are the following

$$\varphi(a,b)=\min\{a,b\},\qquad \varphi(a,b)=\sqrt{a^2+b^2}-a-b,$$

which are called the *Minimum function* [6] and the *Fischer–Burmeister function* [7], respectively.

Other example of complementarity functions is the family φ_λ introduced in [7] and defined by

$$\varphi_\lambda(a,b)=\sqrt{(a-b)^2+\lambda ab}-a-b, \tag{3}$$

where $\lambda\in(0,4)$.

It is important to observe that the *Minimum function* [6] and the *Fischer–Burmeister function* are particular cases of the family φ_λ.

The nonsmooth system of nonlinear equations (2), equivalently the NCP, has been solved using nonsmooth methods of *Newton* [8] and quasi-*Newton* [9, 10, 11, 12] type, and smooth methods [13, 14, 15]. These methods are based on *Clarke*'s generalized Jacobian [16] defined by a *Lipschitz* continuous function $G:\mathbb{R}^n\to\mathbb{R}^n$ as follows

$$\partial G(\boldsymbol{x})=\text{hull}\left\{\lim_{k\to\infty}F'(\boldsymbol{x}_k)\in\mathbb{R}^{n\times n}:\boldsymbol{x}_k\to\boldsymbol{x},\ \boldsymbol{x}_k\in D_G\right\},$$

where D_G denotes the set of all points where G is differentiable and $\text{hull}(A)$ is the convex envelope of A. In general, this set is difficult to compute [17]; in this paper, we use the overestimation given by [18, Proposition 2.6.2 (e)],

$$\partial G(\boldsymbol{x})^T\subseteq\partial G_1(\boldsymbol{x})\times\ldots\times\partial G_n(\boldsymbol{x})\equiv\partial_C G(\boldsymbol{x})^T, \tag{4}$$

where the right-hand side (often easier to compute [19]) denotes the set of matrices in $\mathbb{R}^{n\times n}$ whose i-th column is given by the generalized gradient of the i-th component function G_i. The set $\partial_C G(\boldsymbol{x})$ is called the *C-subdifferential* of G at $\boldsymbol{x}$.

The nonsmooth *Newton* methods [8, 19] solve at each iteration the generalized *Newton* equation

$$H_k \boldsymbol{s}_k = -\Phi(\boldsymbol{x}_k), \tag{5}$$

where $H_k \in \partial\Phi(\boldsymbol{x}_k)$, or $H_k \in \partial_C \Phi(\boldsymbol{x}_k)$.

A way to deal with the nonsmoothness of Φ and to solve (2) is to use a *Jacobian smoothing* method introduced in [20]. The basic idea of this type of methods is to approximate Φ by a smooth operator $\Phi_\mu : \mathbb{R}^n \to \mathbb{R}^n$, where $\mu > 0$ denotes the smoothing parameter, and then to solve a sequence of problems

$$\Phi_\mu(\boldsymbol{x}) = 0, \tag{6}$$

forcing μ to go to zero. For this, a Jacobian smoothing method tries to solve at each iteration the mixed *Newton* equation

$$\Phi'_\mu(\boldsymbol{x}_k) s_k = -\Phi(\boldsymbol{x}_k), \tag{7}$$

where $\Phi'_\mu(\boldsymbol{x}_k)$ is the Jacobian matrix of the function Φ_μ at $\boldsymbol{x}_k$. The linear system (7) uses the unperturbed right-hand side of equation (5), but it replaces the $H_k \in \partial\Phi_\lambda(\boldsymbol{x}_k)$ by a suitable approximation $\Phi'_\mu(x_k)$.

The authors in [20] developed the convergence theory of Jacobian smoothing methods for a special type of functions. A new algorithm for general functions was presented in [17], where the authors use the strategy of JAcobian smoothing to solve the NCP by reformulating it as a system of non-linear equations using the *Fischer-Burmeister* complementarity function.

The good results obtained in [17] and the fact that the family of functions (3) has not been used in connection with the Jacobian smoothing method motivated us to use that strategy to solve the NCP through its reformulation as a non-differentiable non-linear system, using a one-parameter family of complementarity functions (3), which we have analyzed and used recently in [21, 22]. That is, we consider the nonsmooth nonlinear system of equations

$$\Phi_\lambda(\boldsymbol{x}) = \begin{pmatrix} \varphi_\lambda(x_1, F_1(\boldsymbol{x})) \\ \vdots \\ \varphi_\lambda(x_n, F_n(\boldsymbol{x})) \end{pmatrix} = 0. \tag{8}$$

The function Φ_λ is locally *Lipschitz* continuous because of the *Lipschitz* continuity of φ_λ (see [23]). Therefore, $\partial\Phi_\lambda(\boldsymbol{x})$ exists.

In this paper, we propose and analyze a smoothing of the family of nonlinear complementarity functions proposed in [7] and we use its properties in combination with the smooth Jacobian strategy used in [17] to present a new generalized Newton-type algorithm to solve a nonsmooth system of equations equivalent to the Nonlinear Complementarity Problem. In addition, we prove that the algorithm converges locally and q-quadratically, analyzing also its numerical performance.

The organization of this paper is as follows: In Section 2, we present a smoothing of a one-parameter family of complementarity functions (3), and analyze its properties. In Section 3, we present a new algorithm that uses the Jacobian smoothing strategy and develop its local convergence theory. In Section 4, we present numerical experiments which permit us to analyze the performance of the proposed algorithm. Finally, in Section 5, we present our concluding remarks.

Smoothing a family of complementarity functions

In this section, we define and analyze a smooth approximation of the family of complementarity functions (3) introduced in [7], which was redefined and analyzed in detail in [23]. Following this reference, we use the $G_\lambda(a,b)$ notation for the first term on the right side of (3). That is, $G_\lambda(a,b) = \sqrt{(a-b)^2+\lambda ab}$.

Definition 2.1. The function $\varphi_{\lambda\mu}$ given by

$$\varphi_{\lambda\mu}(a,b)=\sqrt{(a-b)^2+\lambda ab+(4-\lambda)\mu}-a-b,\ \lambda\in(0,4),\ \mu>0 \quad (9)$$

is a smooth approximation of a family of complementarity functions φ_λ.

The function $\varphi_{\lambda\mu}$ is well defined for all $\lambda\in(0,4)$ and for all $\mu>0$. Indeed,

$$(a-b)^2+\lambda ab+(4-\lambda)\mu=\begin{pmatrix}a & b\end{pmatrix}K\begin{pmatrix}a\\ b\end{pmatrix}+(4-\lambda)\mu>0,$$

where

$$K=\frac{1}{2}\begin{pmatrix}2 & \lambda-2\\ \lambda-2 & 2\end{pmatrix} \quad (10)$$

is a symmetric and positive definite matrix [23].

The following lemma guarantees that $\varphi_{\lambda\mu}$ is a "perturbed" complementarity function.

Lemma 2.2. *The function* $\varphi_{\lambda\mu}$ *satisfies the following equivalence*

$$\varphi_{\lambda\mu}(a,b)=0 \iff a\geq 0,\ b\geq 0,\ ab=\mu.$$

Proof. It is a direct consequence of the definition of $\varphi_{\lambda\mu}$. □

A useful bound for later theoretical developments is given in the following lemma.

Lemma 2.3. *Lets* $\lambda\in(0,4)$ *and* $\mu>0$. *The function* $G_{\lambda\mu}$, *defined for all* $(a,b)\in\mathbb{R}^2$ *by*

$$G_{\lambda\mu}(a,b)=\sqrt{(a-b)^2+\lambda ab+(4-\lambda)\mu}, \tag{11}$$

satisfies the following inequality

$$G_{\lambda\mu}(a,b)\ \geq\ \sqrt{\alpha_{\min}}\ \left\|\begin{pmatrix}a\\ b\end{pmatrix}\right\|_2, \tag{12}$$

where $\alpha_{\min}>0$ *is the smallest eigenvalue of the matrix* K *defined by* (10).

Proof. Let $\alpha_{\min}>0$ be the smallest eigenvalue of the matrix K in (10), then

$$\begin{aligned}G_{\lambda\mu}(a,b)&=\sqrt{(a-b)^2+\lambda ab+(4-\lambda)\mu}\geq\sqrt{(a-b)^2+\lambda ab}\\ &\geq\sqrt{\alpha_{\min}}\ \left\|\begin{pmatrix}a\\ b\end{pmatrix}\right\|_2,\end{aligned}$$

where the last inequality is given by [23, Lemma 1]. □

Observe that the function φ_λ is nondifferentiable at $(0,0)$ but its smoothing function $\varphi_{\lambda\mu}$ is differentiable, and its gradient vector is defined by

$$\nabla\varphi_{\lambda\mu}(a,b)=\begin{pmatrix}\dfrac{2(a-b)+\lambda b}{2G_{\lambda\mu}(a,b)}-1\\[2ex] \dfrac{-2(a-b)+\lambda a}{2G_{\lambda\mu}(a,b)}-1\end{pmatrix}=\nabla G_{\lambda\mu}(a,b)-\begin{pmatrix}1\\ 1\end{pmatrix}. \tag{13}$$

For later use, we will denote the partial derivatives of $G_{\lambda\mu}$ as follows

$$\alpha_{\lambda\mu}(a,b)=\frac{2(a-b)+\lambda b}{2G_{\lambda\mu}(a,b)},\quad\text{and}\quad\beta_{\lambda\mu}(a,b)=\frac{-2(a-b)+\lambda a}{2G_{\lambda\mu}(a,b)}, \tag{14}$$

and the derivatives of G_λ by

$$\alpha_{\lambda}(a,b)=\frac{2(a-b)+\lambda b}{2G_{\lambda}(a,b)},\quad\text{and}\quad\beta_{\lambda}(a,b)=\frac{-2(a-b)+\lambda a}{2G_{\lambda}(a,b)}. \tag{15}$$

From $G_{\lambda\mu}(a,b) \geq G_{\lambda}(a,b)$ and [7, Lemma 2.4], we have

$$||\nabla G_{\lambda\mu}(a,b)||_2 \leq \sqrt{2}. \tag{16}$$

From (13), (16) and the triangle inequality, we have

$$||\nabla \varphi_{\lambda\mu}(a,b)||_2 \leq 2\sqrt{2}. \tag{17}$$

Following [23], we have a matrix expression for $\nabla G_{\lambda\mu}(a,b)$ analogous to the one given for $\nabla G_{\lambda}(a,b)$, namely

$$\nabla G_{\lambda\mu}(a,b) = \frac{1}{G_{\lambda\mu}(a,b)} K \begin{pmatrix} a \\ b \end{pmatrix}, \tag{18}$$

where K is the matrix given by (10) which satisfies $||K||_2 < 2$.

An important property of the smoothing function $\varphi_{\lambda\mu}$ is that it is an uniformly continuous function; moreover, it is *Lipschitz continuous* as guaranteed by the following lemma.

Lemma 2.4. *The function $\varphi_{\lambda\mu}$ is Lipschitz continuous with constant $2\sqrt{2}$. That is, for all $x, y \in \mathbb{R}^2$*

$$|\varphi_{\lambda\mu}(x) - \varphi_{\lambda\mu}(y)| \leq 2\sqrt{2}\, ||x-y||_2 .$$

Proof. Let x, $y \in \mathbb{R}^2$. By the Mean Value Theorem, there exists a vector z in (x,y) such that,

$$\varphi_{\lambda\mu}(x) - \varphi_{\lambda\mu}(y) = \nabla \varphi_{\lambda\mu}(z)^T (x-y).$$

using a *Cauchy-Schwartz* inequality and the bound (17), we have

$$|\varphi_{\lambda\mu}(x) - \varphi_{\lambda\mu}(y)| \leq ||\nabla \varphi_{\lambda\mu}(z)||_2 ||x-y||_2 \leq 2\sqrt{2}\, ||x-y||_2$$

thus, we conclude that $\varphi_{\lambda\mu}$ is *Lipschitz* continuous with constant $2\sqrt{2}$. □

Corollary 2.5. *The function $G_{\lambda\mu}$ is Lipschitz continuous, that is, for all possible $x, y \in \mathbb{R}^2$, we have the following bound*

$$|G_{\lambda\mu}(x) - G_{\lambda\mu}(y)| \leq \sqrt{2}||x-y||_2. \tag{19}$$

Proof. It is analogous to the proof of the previous lemma. □

The following lemma guarantees that $\nabla \varphi_{\lambda\mu}$ is also a locally *Lipschitz* continuous function.

Lemma 2.6. *Let* $\varphi_{\lambda\mu}$ *the function defined by* (9)*,* $\boldsymbol{w}$ *a nonzero vector in* $\mathbb{R}^2$ *and* $\mathscr{B}(\boldsymbol{w};\epsilon)$ *a ball with* $0<\epsilon<\frac{1}{2}\|\boldsymbol{w}\|_2$*. Then there exists* $\eta>0$ *such that for all* $\boldsymbol{u},\boldsymbol{v}\in\mathscr{B}(\boldsymbol{w};\epsilon)$*,*

$$\left\|\nabla\varphi_{\lambda\mu}(\boldsymbol{u})-\nabla\varphi_{\lambda\mu}(\boldsymbol{v})\right\|_2\leq\eta\|\boldsymbol{u}-\boldsymbol{v}\|_2. \tag{20}$$

Proof. It is similar to the proof of [23, Lemma 5] using (12), (16), (18), and the Corollary 2.5. □

Lemma 2.7. *Let* φ_{λ} *and* $\varphi_{\lambda\mu}$ *the functions defined by* (3) *and* (9)*, respectively;* $\boldsymbol{w}$ *a nonzero vector in* $\mathbb{R}^2$ *and* $\mathscr{B}(\boldsymbol{w};\epsilon)$*, with* $0<\epsilon<\frac{1}{2}\|\boldsymbol{w}\|_2$*, a ball that does not contain* $(0,0)$*. Then there exists* $\tau>0$ *such that, for all* $\boldsymbol{u},\boldsymbol{v}\in\mathscr{B}(\boldsymbol{w};\epsilon)$*,*

$$\left\|\nabla\varphi_{\lambda\mu}(\boldsymbol{u})-\nabla\varphi_{\lambda}(\boldsymbol{v})\right\|_2\leq\tau\|\boldsymbol{u}-\boldsymbol{v}\|_2. \tag{21}$$

Proof. It is analogous to the proof of [23, Lemma 5] taking into account the inequality $G_\lambda(c,d)\leq G_{\lambda\mu}(c,d)$ and therefore,

$$\left|\frac{1}{G_{\lambda\mu}(a,b)}-\frac{1}{G_\lambda(c,d)}\right|=\left|\frac{G_\lambda(c,d)-G_{\lambda\mu}(a,b)}{G_{\lambda\mu}(a,b)G_\lambda(c,d)}\right|\leq\left|\frac{G_{\lambda\mu}(c,d)-G_{\lambda\mu}(a,b)}{G_{\lambda\mu}(a,b)G_\lambda(c,d)}\right|.$$

□

Algorithm and convergence theory

Following the idea of Jacobian smoothing methods presented in the introduction of this paper, we propose the following basic algorithm for solving the nonsmooth nonlinear system of equations (8).

Algorithm 3.1. Given x_0, an initial point, $\lambda\in(0,4)$, and $\{\mu_k\}$ a sequence that converges to zero, for $k=1,2,\ldots,$ compute

$$\begin{aligned}\Phi'_{\lambda\mu}(\boldsymbol{x}_k)\boldsymbol{s}_k&=-\Phi_\lambda(\boldsymbol{x}_k),\\ \boldsymbol{x}_{k+1}&=\boldsymbol{x}_k+\boldsymbol{s}_k,\end{aligned} \tag{22}$$

where $\Phi'_{\lambda\mu}(\boldsymbol{x}_k)$ is the Jacobian matrix of the function $\Phi_{\lambda\mu}$ at $\boldsymbol{x}_k$.

It is important to observe that we use the matrix $\Phi'_{\lambda\mu}(\boldsymbol{x}_k)$ instead of $H_k\in\partial_C\Phi_\lambda(\boldsymbol{x}_k)$.

The first result characterizes the matrices of the *C-subdifferential* of Φ_λ at $\boldsymbol{x}$.

Proposition 3.2. *For an arbitrary* $\boldsymbol{x}\in\mathbb{R}^n$*, we have*

$$\partial_C G(\boldsymbol{x})^T=D_\alpha(\boldsymbol{x})+F'(\boldsymbol{x})^T D_\beta(\boldsymbol{x}),$$

where $D_\alpha(\boldsymbol{x}) = \mathrm{diag}(\alpha_1(\boldsymbol{x}), \ldots, \alpha_n(\boldsymbol{x}))$ *and* $D_\beta(\boldsymbol{x}) = \mathrm{diag}(\beta_1(\boldsymbol{x}), \ldots, \beta_n(\boldsymbol{x}))$ *are diagonal matrices whose* i*-th diagonal element is given by*

$$\alpha_i(\boldsymbol{x}) = \alpha_\lambda(x_i, F_i(\boldsymbol{x})) - 1, \qquad \beta_i(\boldsymbol{x}) = \beta_\lambda(x_i, F_i(\boldsymbol{x})) - 1.$$

if $(x_i, F_i(\boldsymbol{x})) \neq (0,0)$, *and by*

$$\alpha_i(\boldsymbol{x}) = \xi_i - 1, \qquad \beta_i(\boldsymbol{x}) = \chi_i - 1,$$

for every $(\xi_i, \chi_i) \in \mathbb{R}^2$ *such that* $\|(\xi_i, \chi_i)\| \leq 2 - \dfrac{\lambda(4-\lambda)}{8}$, *in the case* $(x_i, F_i(\boldsymbol{x})) = (0,0)$.

Proof. It follows directly from the definition of the *C-subdifferential* and from [7, Proposition 2.5]. □

By the differentiabilty of $\Phi_{\lambda\mu}$ at $\boldsymbol{x}_k$, the Jacobian matrix $\Phi'_{\lambda\mu}(\boldsymbol{x}_k)$ is given by

$$\Phi'_{\lambda\mu}(\boldsymbol{x}_k) = \begin{pmatrix} \nabla\varphi_{\lambda\mu}(x_1^k, F_1(\boldsymbol{x}_k))^T \\ \vdots \\ \nabla\varphi_{\lambda\mu}(x_n^k, F_n(\boldsymbol{x}_k))^T \end{pmatrix},$$

where its i-th row $\nabla\varphi_{\lambda\mu}(x_i^k, F_i(\boldsymbol{x}_k))^T$, denoted by $[\Phi'_{\lambda\mu}(\boldsymbol{x}_k)]_i$, has the form

$$[\Phi'_{\lambda\mu}(\boldsymbol{x}_k)]_i = \Big(\alpha_{\lambda\mu}(x_i^k, F_i(\boldsymbol{x}_k)) - 1\Big)\mathbf{e}_i^T + \Big(\beta_{\lambda\mu}(x_i^k, F_i(\boldsymbol{x}_k)) - 1\Big)\nabla F_i(\boldsymbol{x}_k)^T,$$

with $\mathbf{e}_1, \ldots, \mathbf{e}_n$, the canonical vectors in $\mathbb{R}^n$, and where $\alpha_{\lambda\mu}$ and $\beta_{\lambda\mu}$ are the functions defined by (14), that is

$$\alpha_{\lambda\mu}(x_i^k, F_i(\boldsymbol{x}_k)) = \frac{2(x_i^k - F_i(\boldsymbol{x}_k)) + \lambda F_i(\boldsymbol{x}_k)}{2\sqrt{(x_i^k - F_i(\boldsymbol{x}_k))^2 + \lambda x_i F_i(\boldsymbol{x}_k) + (4-\lambda)\mu}},$$

$$\beta_{\lambda\mu}(x_i^k, F_i(\boldsymbol{x}_k)) = \frac{-2(x_i^k - F_i(\boldsymbol{x}_k)) + \lambda x_i^k}{2\sqrt{(x_i^k - F_i(\boldsymbol{x}_k))^2 + \lambda x_i F_i(\boldsymbol{x}_k) + (4-\lambda)\mu}}.$$

The next lemma guarantees that, as the parameter μ tends to zero, the distance between the matrix $\Phi'_{\lambda\mu}(\boldsymbol{x})$ and the set $\partial_C \Phi'_\lambda(\boldsymbol{x})$ also tends to zero. Therefore, it is reasonable to replace the generalized *Newton* iteration (5) by the iteration (22).

Lemma 3.3. *Let* $\boldsymbol{x} \in \mathbb{R}^n$ *be arbitrary but fixed and* $\mu > 0$. *Then we have*

$$\lim_{\mu \downarrow 0} \mathrm{dist}\Big(\Phi'_{\lambda\mu}(\boldsymbol{x}), \partial_C \Phi_\lambda(\boldsymbol{x})\Big) = 0. \tag{23}$$

Proof. Since

$$\lim_{\mu\downarrow 0} \operatorname{dist}\left(\Phi'_{\lambda\mu}(\boldsymbol{x}), \partial_C \Phi_\lambda(\boldsymbol{x})\right) = \lim_{\mu\downarrow 0} \inf_{H\in\partial_C\Phi_\lambda(\boldsymbol{x})} \left\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H\right\|_F$$
$$= \inf_{H\in\partial_C\Phi_\lambda(\boldsymbol{x})} \lim_{\mu\downarrow 0} \left\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H\right\|_F,$$

in order to prove (23) it is enough to prove that is the limit of $\Phi'_{\lambda\mu}(\boldsymbol{x})$ when $\mu \to 0$ is in $\partial_C \Phi_\lambda(\boldsymbol{x})$. To do this, let us define the index set $\Gamma(\boldsymbol{x}) = \{i : x_i = F_i(\boldsymbol{x}) = 0\}$. We have

$$\lim_{\mu\downarrow 0} [\Phi'_{\lambda\mu}(\boldsymbol{x})]_i$$
$$= \begin{cases} (\alpha_\lambda(x_i, F_i(\boldsymbol{x})) - 1)\boldsymbol{e}_i^T + (\beta_\lambda(x_i, F_i(\boldsymbol{x})) - 1)\nabla F_i(\boldsymbol{x})^T, & i \notin \Gamma(\boldsymbol{x}) \\ -\boldsymbol{e}^T - \nabla F_i(\boldsymbol{x})^T, & i \in \Gamma(\boldsymbol{x}) \end{cases}$$
$$= [H]_i,$$

then the matrix H has the form described in Proposition 1 with $(\xi_i, \chi_i) = (0,0)$, for $i \in \Gamma(\boldsymbol{x})$, therefore, $H \in \partial_C \Phi_\lambda(\boldsymbol{x})$. Thus, the infimun is zero and (23) is satisfied. □

The following hypotheses allow to prove that the Algorithm 3.1 is well defined and converges to a solution of (8).

H1. The system (8) has a solution $\boldsymbol{x}_* \in \mathbb{R}^n$.

H2. The Jacobian matrix of F is locally *Lipschitz continuous*.

H3. The matrices of $\partial_C \Phi_\lambda(\boldsymbol{x}_*)$ are nonsingular.

By the compactness of $\partial \Phi_\lambda(\boldsymbol{x}_*)$ (cf. [18]) and from H3, there is a constant β such that

$$||H_*^{-1}|| \leq \beta, \tag{24}$$

for all $H_* \in \partial_C \Phi_\lambda(\boldsymbol{x}_*)$.

Next, we present a technical lemma that will be useful in the proof of Lemma 3.5.

Lemma 3.4. *Assume* H1 *and* H3, *and* $r \in (0, 1)$. *There exists a positive constant* ϵ *such that, if* $||\boldsymbol{x} - \boldsymbol{x}_*||_\infty < \epsilon$ *then the function* $\mathcal{Q}$ *defined by*

$$\mathcal{Q}(\boldsymbol{x}) = \boldsymbol{x} - \Phi'_{\lambda\mu}(\boldsymbol{x})^{-1}\Phi_\lambda(\boldsymbol{x}), \tag{25}$$

is well defined, and

$$||\mathcal{Q}(\boldsymbol{x}) - \boldsymbol{x}_*||_\infty \leq r\, ||\boldsymbol{x} - \boldsymbol{x}_*||_\infty. \tag{26}$$

Moreover, if the Jacobian matrix of F *satisfies Assumption* H2 *then*

$$\|\mathscr{Q}(\boldsymbol{x})-\boldsymbol{x}_*\|_\infty \leq c\,\|\boldsymbol{x}-\boldsymbol{x}_*\|_\infty^2. \tag{27}$$

Proof. In order to prove that $\Phi'_{\lambda\mu}(\boldsymbol{x})$ is nonsingular, we use the *Banach's Lemma* [14]. To do this, we find a bound for $\|\Phi'_{\lambda\mu}(\boldsymbol{x})-H_*\|$, where $H_* \in \partial_C \Phi_\lambda(\boldsymbol{x}_*)$. Using the definition of *C-subdifferential*, we have that the matrix H_* has the form

$$H_* = \begin{pmatrix} [H_*]_1 \\ \vdots \\ [H_*]_n \end{pmatrix} = \lim_{k\to\infty} \Phi'_\lambda(\boldsymbol{y}^k) = \begin{pmatrix} \lim\limits_{k\to\infty} \nabla\varphi_\lambda\left(y_1^k, F_1(\boldsymbol{y}^k)\right)^T \\ \vdots \\ \lim\limits_{k\to\infty} \nabla\varphi_\lambda\left(y_n^k, F_n(\boldsymbol{y}^k)\right)^T \end{pmatrix},$$

where the sequence $\{\boldsymbol{y}^k\}$ converges to $\boldsymbol{x}_*$ and satisfies that $\Phi'_\lambda(\boldsymbol{y}^k)$ exists.

We bound $\|\Phi'_{\lambda\mu}(\boldsymbol{x})-H_*\|$ using the definition of infinite matrix norm. Thus, for some $j \in \{1,2,\dots,n\}$, we have

$$\begin{aligned}\|\Phi'_{\lambda\mu}(\boldsymbol{x})-H_*\|_\infty &= \left\|\left[\Phi'_{\lambda\mu}(\boldsymbol{x}_*)\right]_j - [H_*]_j\right\|_1 \\ &\leq n\left\|\nabla\varphi_{\lambda\mu}(x_j, F_j(\boldsymbol{x}))^T - \lim_{k\to\infty}\nabla\varphi_\lambda\left(y_j^k, F_j(\boldsymbol{y}^k)\right)^T\right\|_\infty \\ &\leq n\lim_{k\to\infty}\left\|\nabla\varphi_{\lambda\mu}(x_j, F_j(\boldsymbol{x}))^T - \nabla\varphi_\lambda\left(y_j^k, F_j(\boldsymbol{y}^k)\right)^T\right\|_\infty.\end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned}\|\Phi'_{\lambda\mu}(\boldsymbol{x})-H_*\|_\infty &\leq n\,\tau\lim_{k\to\infty}\left\|\begin{pmatrix} x_j - y_j^k \\ F_j(\boldsymbol{x})-F_j(\boldsymbol{y}^k)\end{pmatrix}\right\| \\ &\leq \sqrt{2}\,n\,\tau\left\|\begin{pmatrix} x_j - x_j^* \\ F_j(\boldsymbol{x})-F_j(\boldsymbol{x}_*)\end{pmatrix}\right\|_\infty \\ &\leq \sqrt{2}\,n\,\tau\max\left\{|x_j-x_j^*|, |F_j(\boldsymbol{x})-F_j(\boldsymbol{x}_*)|\right\}.\end{aligned} \tag{28}$$

By de continuity of F, for all $\hat{\epsilon}$, there exists $\hat{\delta} > 0$ such that, if we have $\|\boldsymbol{x}-\boldsymbol{x}^*\|_\infty < \hat{\delta}$ then $|F_j(\boldsymbol{x})-F_j(\boldsymbol{x}^*)| < \hat{\epsilon}$. Let $\epsilon' = \min\{\hat{\epsilon}, \hat{\delta}\}$. We consider the two possibilities for the maximum in (28).

1. If $\max\left\{|x_j-x_j^*|, |F_j(\boldsymbol{x})-F_j(\boldsymbol{x}_*)|\right\} = |x_j - x_j^*| \leq \|\boldsymbol{x}-\boldsymbol{x}_*\|_\infty \leq \epsilon' < \hat{\epsilon}$.
2. If $\max\left\{|x_j-x_j^*|, |F_j(\boldsymbol{x})-F_j(\boldsymbol{x}_*)|\right\} = |F_j(\boldsymbol{x}^k)-F_j(\boldsymbol{x}_*)| < \hat{\epsilon}$.

Thus, from (28), for any $\hat{\epsilon}$

$$\|\Phi'_{\lambda\mu}(x) - H_*\|_\infty < \sqrt{2}\, n\, \tau\, \hat{\epsilon}\cdot$$

Let $\hat{\epsilon} < \dfrac{1}{2\sqrt{2}\rho n \beta}$, then $\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H_*\|_\infty \leq \tau n \hat{\epsilon} < \dfrac{1}{2\beta}\cdot$ Now,

$$\|H_*^{-1}\Phi'_{\lambda\mu}(\boldsymbol{x}) - I\|_\infty \leq \|H_*^{-1}\|_\infty \|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H_*\|_\infty \leq \beta\frac{1}{2\beta} = \frac{1}{2},$$

thus $\|H_*^{-1}\Phi'_{\lambda\mu}(\boldsymbol{x}) - I\|_\infty < 1$; therefore, there exists $\Phi'_{\lambda\mu}(\boldsymbol{x})^{-1}$ (*Banach's Lemma* [14]) and the function $\mathcal{Q}$ is well defined. In addition,

$$\|\Phi'_{\lambda\mu}(\boldsymbol{x})^{-1}\|_\infty \leq \frac{\|H_*^{-1}\|_\infty}{1 - \|H_*^{-1}\Phi'_{\lambda\mu}(\boldsymbol{x}) - I_n\|_\infty} \leq \frac{\beta}{1 - \frac{1}{2}} = 2\beta.$$

In the second part of the proof, we prove (26). For this, subtracting $\boldsymbol{x}_*$ in (25), using $\|\cdot\|_\infty$, and performing some algebraic manipulations, we have

$$\begin{aligned}
\|\mathcal{Q}(\boldsymbol{x}) - \boldsymbol{x}_*\|_\infty &= \|(\boldsymbol{x} - \boldsymbol{x}_*) - \Phi'_{\lambda\mu}(\boldsymbol{x})^{-1}\Phi_\lambda(\boldsymbol{x})\|_\infty \\
&= \|\Phi'_{\lambda\mu}(\boldsymbol{x})^{-1}[\Phi'_{\lambda\mu}(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{x}_*) - \Phi_\lambda(\boldsymbol{x}) + \Phi_\lambda(\boldsymbol{x}_*)]\|_\infty \qquad (29) \\
&\leq \|\Phi'_{\lambda\mu}(\boldsymbol{x})^{-1}\|_\infty \|\Phi'_{\lambda\mu}(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{x}_*) - \Phi_\lambda(\boldsymbol{x}) + \Phi_\lambda(\boldsymbol{x}_*)\|_\infty \\
&\leq 2\beta\, \|\Phi'_{\lambda\mu}(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{x}_*) - H(\boldsymbol{x} - \boldsymbol{x}_*) + H(\boldsymbol{x} - \boldsymbol{x}_*) - \Phi_\lambda(\boldsymbol{x}) + \Phi_\lambda(\boldsymbol{x}_*)\|_\infty \\
&\leq 2\beta\left[\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H\|\, \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty + \|H(\boldsymbol{x} - \boldsymbol{x}_*) - \Phi_\lambda(\boldsymbol{x}) + \Phi_\lambda(\boldsymbol{x}_*)\|_\infty\right]
\end{aligned}$$

By Lemma 3.3, for all $\delta > 0$, there exists $\epsilon_1 > 0$ such that, if $\|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty < \epsilon_1$,

$$\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H\| < \delta.$$

thus, for $\delta < \dfrac{r}{4\beta}$, there exists $\epsilon_r > 0$ such that, if $\|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty < \epsilon_r$,

$$\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H\| < \frac{r}{4\beta}\cdot \qquad (30)$$

Moreover, by Theorem 2.3 in [7], for all $\rho > 0$, there exists $\epsilon_2 > 0$, such that, if $\|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty < \epsilon_1$ then $\|H(\boldsymbol{x} - \boldsymbol{x}_*) - \Phi_\lambda(\boldsymbol{x}) + \Phi_\lambda(\boldsymbol{x}_*)\|_\infty < \rho\, \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty$. In particular, for $\rho < \frac{r}{4\beta}$, there exists $\bar{\epsilon}_r > 0$ such that, if $\|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty < \bar{\epsilon}_r$ then

$$\|H(\boldsymbol{x} - \boldsymbol{x}_*) - \Phi_\lambda(\boldsymbol{x}) + \Phi_\lambda(\boldsymbol{x}_*)\|_\infty < \frac{r}{4\beta}\, \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty\,. \qquad (31)$$

Thus, for $\epsilon = \min\{\epsilon', \epsilon_r, \bar{\epsilon}_r\}$, we have (29), (30), and (31), therefore

$$\|\mathscr{Q}(\boldsymbol{x}) - \boldsymbol{x}_*\|_\infty \leq r\, \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty,$$

and we have proved (26).

The third part of the proof consists in obtaining (27) under the hypothesis H2. By [7, Theorem 2.3], there exists a positive constant γ such that

$$\|H(\boldsymbol{x} - \boldsymbol{x}_*) - \Phi_\lambda(\boldsymbol{x}) + \Phi_\lambda(\boldsymbol{x}_*)\|_\infty \leq \gamma\, \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty^2. \tag{32}$$

On the other hand, adding and subtracting H_* in $\left\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H\right\|_\infty$ and using the triangle inequality, we have

$$\left\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H\right\|_\infty \leq \left\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H_*\right\|_\infty + \left\|H_* - H\right\|_\infty. \tag{33}$$

By (28), a bound for the first term of (33) is $\sqrt{2}\, n\, \tau\, M$, where

$$M = \max\left\{|x_j - x_j^*|, |F_j(\boldsymbol{x}) - F_j(\boldsymbol{x}_*)|\right\},$$

and by [23, Lemma 4.2] the second term is bounded by $\sqrt{2}\, n\, \eta\, M$, where η is the *Lipschitz* constant of $\nabla\varphi_\lambda$. Then,

$$\left\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H\right\|_\infty \leq \sqrt{2}\, n\, (\tau + \eta) M. \tag{34}$$

Moreover,

1. If $M = |x_j - x_j^*|$ then $M \leq \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty$.
2. If $M = |F_j(\boldsymbol{x}_k) - F_j(\boldsymbol{x}_*)|$ then $M \leq \|F(\boldsymbol{x}_k) - F(\boldsymbol{x}_*)\|_\infty \leq \zeta \|\boldsymbol{x}_k - \boldsymbol{x}_*\|_\infty$,

where the last inequality is given by [14, Lemma 4.1.16], because F is continuously differentiable and its Jacobian matrix is *Lipschitz* continuous in a neighborhood of $\boldsymbol{x}_*$.

Let $\bar{\zeta} = \max\{1, \zeta\}$. From (34) and the previous cases about M,

$$\left\|\Phi'_{\lambda\mu}(\boldsymbol{x}) - H\right\|_\infty \leq \sqrt{2}\, n\, (\tau + \eta)\, \bar{\zeta}\, \|\boldsymbol{x}_k - \boldsymbol{x}_*\|_\infty = \theta \|\boldsymbol{x}_k - \boldsymbol{x}_*\|_\infty, \tag{35}$$

with $\theta = \sqrt{2}\, n\, (\tau + \eta)\, \bar{\zeta}$. Thus, from (29), (32), and (35),

$$\|\mathscr{Q}(\boldsymbol{x}) - \boldsymbol{x}_*\|_\infty \leq 2\beta\left[\theta\, \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty^2 + \gamma\, \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty^2\right] \leq 2\,\beta\,(\theta + \gamma)\, \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty^2.$$

Therefore,

$$\|\mathscr{Q}(\boldsymbol{x}) - \boldsymbol{x}_*\|_\infty \leq c\, \|\boldsymbol{x} - \boldsymbol{x}_*\|_\infty^2,$$

with $c = 2\,\beta\,(\theta + \gamma)$. □

The following lemma guarantees that the proposed algorithm is well defined, it converges linearly and it gives a sufficient condition for q-quadratic convergence.

Lemma 3.5. *There exists a positive* ϵ_0 *such that, if* $||\boldsymbol{x}_0 - \boldsymbol{x}_*||_\infty < \epsilon_0$ *then*

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \Phi'_{\lambda\mu}(\boldsymbol{x}_k)^{-1}\Phi_\lambda(\boldsymbol{x}_k) \tag{36}$$

generates a well defined sequence $\{\boldsymbol{x}_k\}$ *which converges to* $\boldsymbol{x}_*$, *and given* $r \in (0, 1)$, *satisfies*

$$\left\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_*\right\|_\infty \le r\, ||\boldsymbol{x}_k - \boldsymbol{x}_*||_\infty . \tag{37}$$

Moreover, if the Jacobian matrix of F *satisfies the assumption* H2 *then*

$$\left\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_*\right\|_\infty \le c\, ||\boldsymbol{x}_k - \boldsymbol{x}_*||^2_\infty , \tag{38}$$

where c *is the constant of* Lemma 3.4.

Proof. Let $\mathcal{Q}$ be defined in (25). Thus, for $k = 0, 1, \ldots$

$$\boldsymbol{x}_{k+1} = \mathcal{Q}(\boldsymbol{x}_k) = \boldsymbol{x}_k - \Phi'_{\lambda\mu}(\boldsymbol{x}_k)^{-1}\Phi_\lambda(\boldsymbol{x}_k).$$

Let $r \in (0, 1)$ and $\epsilon_0 \in (0, \epsilon)$, where ϵ is the constant of Lemma 3.3. The proof is by induction on k.

- For $k = 0$, if $||\boldsymbol{x}_0 - \boldsymbol{x}_*||_\infty \le \epsilon_0 < \epsilon$, by Lemma 3.3, $\boldsymbol{x}_1 = \mathcal{Q}(\boldsymbol{x}_0)$ is well defined and verifies

  $$||\boldsymbol{x}_1 - \boldsymbol{x}_*||_\infty \le r\, ||\boldsymbol{x}_0 - \boldsymbol{x}_*||_\infty .$$

  Moreover, if the Jacobian matrix of F is *Lipschitz* continuous in a neighborhood of $\boldsymbol{x}_*$,

  $$||\boldsymbol{x}_1 - \boldsymbol{x}_*||_\infty \le c\, ||\boldsymbol{x}_0 - \boldsymbol{x}_*||^2_\infty .$$

- *Induction hypotheses*. For all $0 < k \le m-1$, we have $||\boldsymbol{x}_k - \boldsymbol{x}_*||_\infty < \epsilon_0$. Then, by Lemma 3.3, $\boldsymbol{x}_m = \mathcal{Q}(\boldsymbol{x}_{m-1})$ is well defined and

  $$||\boldsymbol{x}_m - \boldsymbol{x}^*||_\infty = ||\mathcal{Q}(\boldsymbol{x}_{m-1}) - \boldsymbol{x}_*||_\infty \le r\, ||\boldsymbol{x}_{m-1} - \boldsymbol{x}^*||_\infty . \tag{39}$$

  Moreover, $||\boldsymbol{x}_m - \boldsymbol{x}_*||_\infty \le c\, ||\boldsymbol{x}_{m-1} - \boldsymbol{x}_*||^2_\infty$, if the Jacobian matrix of F is *Lipschitz* continuous at $\boldsymbol{x}_*$.

  From (39), we have,

  $$||\boldsymbol{x}_m - \boldsymbol{x}_*||_\infty \le r\, ||\boldsymbol{x}_{m-1} - \boldsymbol{x}_*||_\infty \le r^m\, ||\boldsymbol{x}_0 - \boldsymbol{x}_*||_\infty \le r^m \epsilon_0 < \epsilon,$$

thus, Lemma 3.3 guarantees that $\boldsymbol{x}_{m+1}$ is well defined and satisfies

$$\|\boldsymbol{x}_{m+1}-\boldsymbol{x}_*\|_\infty \leq r\,\|\boldsymbol{x}_m-\boldsymbol{x}_*\|_\infty,$$

and, if H2 is verified then $\boldsymbol{x}_{m+1}$ satisfies

$$\|\boldsymbol{x}_{m+1}-\boldsymbol{x}_*\|_\infty \leq c\,\|\boldsymbol{x}_m-\boldsymbol{x}_*\|_\infty^2.$$

Therefore, we conclude that (37) and (38) are true for all $k=0,1,\ldots$. □

Numerical experiments

In this section, we analyze numerically the Jacobian smoothing algorithm introduced in the previous section (Algorithm 1). To do this, we compared our algorithm with the one proposed in [17] (which is a particular case of our algorithm when $\lambda=2$), which we call Algorithm 2, and with a *Newton* method that uses matrices in the *C-subdifferential* of Φ_λ at x_k.

For the variation of parameter μ, we consider the sequences $\{\mu_0 2^{-k}\}$ and $\{\mu_0 100^{-k}\}$, with $\mu_0=\dfrac{\alpha}{2\varkappa}$, $\varkappa=\sqrt{2n}$, $\alpha\in(0,1)$ given in [17]. Moreover, in both, the proposed algorithm and in the *Newton* method, the parameter λ varies from $\lambda=10^{-1}$ to $\lambda=3.9$ with increments of 10^{-1}. It is important to mention that in the Algorithm 2 proposed in [17], $\lambda=2$ and μ depend on the function F.

We use five test problems, four of them were chosen from [6]. The fifth function is [24, Example A]. We describe each of these functions below, as well as the initial point used (x_0) and the solutions found (x_*).

1. *Billups problem* [6]. Let $F:\mathbb{R}\to\mathbb{R}$ be defined by

$$F(x)=(x-1)^2-1.1,$$

$x_0=0$ and $x_*=2.0488$.

2. *Mathiesen-modified problem* [6]. Let $F:\mathbb{R}^4\to\mathbb{R}^4$ be defined by

$$F(\boldsymbol{x})=\begin{pmatrix} -x_2+x_3+x_4 \\ x_1-\dfrac{4.5x_3+2.7x_4}{x_2+1} \\ 5-x_1-\dfrac{0.5x_3+0.3x_4}{x_3+1} \\ 3-x_1 \end{pmatrix},$$

$x_0=(1\ 1\ 1\ 1)^T$ and $\boldsymbol{x}_*=(a\ 0\ 0\ 0)^T$, with $a\in[0,3]$.

3. *Kojima-Shindo problem* [6]. Let $F:\mathbb{R}^4\rightarrow\mathbb{R}^4$ be defined by

$$F(\boldsymbol{x})=\begin{pmatrix}3x_1^2+2x_1x_2+2x_2^2+x_3+3x_4-6\\ 2x_1^2+x_2^2+x_1+10x_3+2x_4-2\\ 3x_1^2+x_1x_2+2x_2^2+2x_3+9x_4-9\\ x_1^2+3x_2^2+2x_3+3x_4-3\end{pmatrix},$$

$\boldsymbol{x}_0=(1\ 1\ 1\ 1)^T$ and $\boldsymbol{x}_*=\left(\sqrt{6}/2\ 0\ 0\ 1/2\right)^T$.

4. *Kojima-Josephy problem* [6]. $F:\mathbb{R}^4\rightarrow\mathbb{R}^4$ be defined by

$$F(\boldsymbol{x})=\begin{pmatrix}3x_1^2+2x_1x_2+2x_2^2+x_3+3x_4-6\\ 2x_1^2+x_2^2+x_1+3x_3+2x_4-2\\ 3x_1^2+x_1x_2+2x_2^2+2x_3+3x_4-9\\ x_1^2+3x_2^2+2x_3+3x_4-3\end{pmatrix},$$

$\boldsymbol{x}_0=(1\ 1\ 1\ 1)^T$ and $\boldsymbol{x}_*=\left(\sqrt{6}/2\ 0\ 0\ 1/2\right)^T$.

5. *Example A in* [24]. $F:\mathbb{R}^5\rightarrow\mathbb{R}^5$ be defined by

$$F(\boldsymbol{x})=\begin{pmatrix}x_1+x_2x_3x_4x_5/50\\ x_2+x_1x_3x_4x_5/50-3\\ x_3+x_1x_2x_4x_5/50-1\\ x_4+x_1x_2x_3x_5/50-0.5\\ x_5+x_1x_2x_3x_4/50\end{pmatrix},$$

$\boldsymbol{x}_0=(1\ -1\ 2\ -2\ 5)^T$ and $\boldsymbol{x}_*=(0\ 3\ 1\ 0\ 0)^T$.

To write the codes of the algorithms and the test functions, we used the *software* MATLAB® and we performed the numerical tests on a computer with a processor Intel(R) Core(TM) i5-4200M CPU @2.50 GHZ. We used as convergence criterion $||\Phi_\lambda(x_k)||<10^{-6}$ or $||\Phi_{\lambda\mu}(x_k)||<10^{-6}$, according to the case. We declared divergence if the number of iterations exceeded 500.

For each problem and with each algorithm, we compared the number of iterations and the time (in seconds) used by the algorithm to obtain convergence for each value of λ. The results obtained are presented below in ten graphics.

Fig. 1 shows that the *Newton* method converges only for $\lambda\geq 3.3$, with more iterations than the proposed algorithm, which converges for $\lambda\geq 2$, and the number of iterations is the same for the two sequences $\{\mu_k\}$ used, therefore its graphics match (Fig. 1A). On the other hand, Algorithm 2 diverges. In addition, Algorithm 1 is the most efficient.

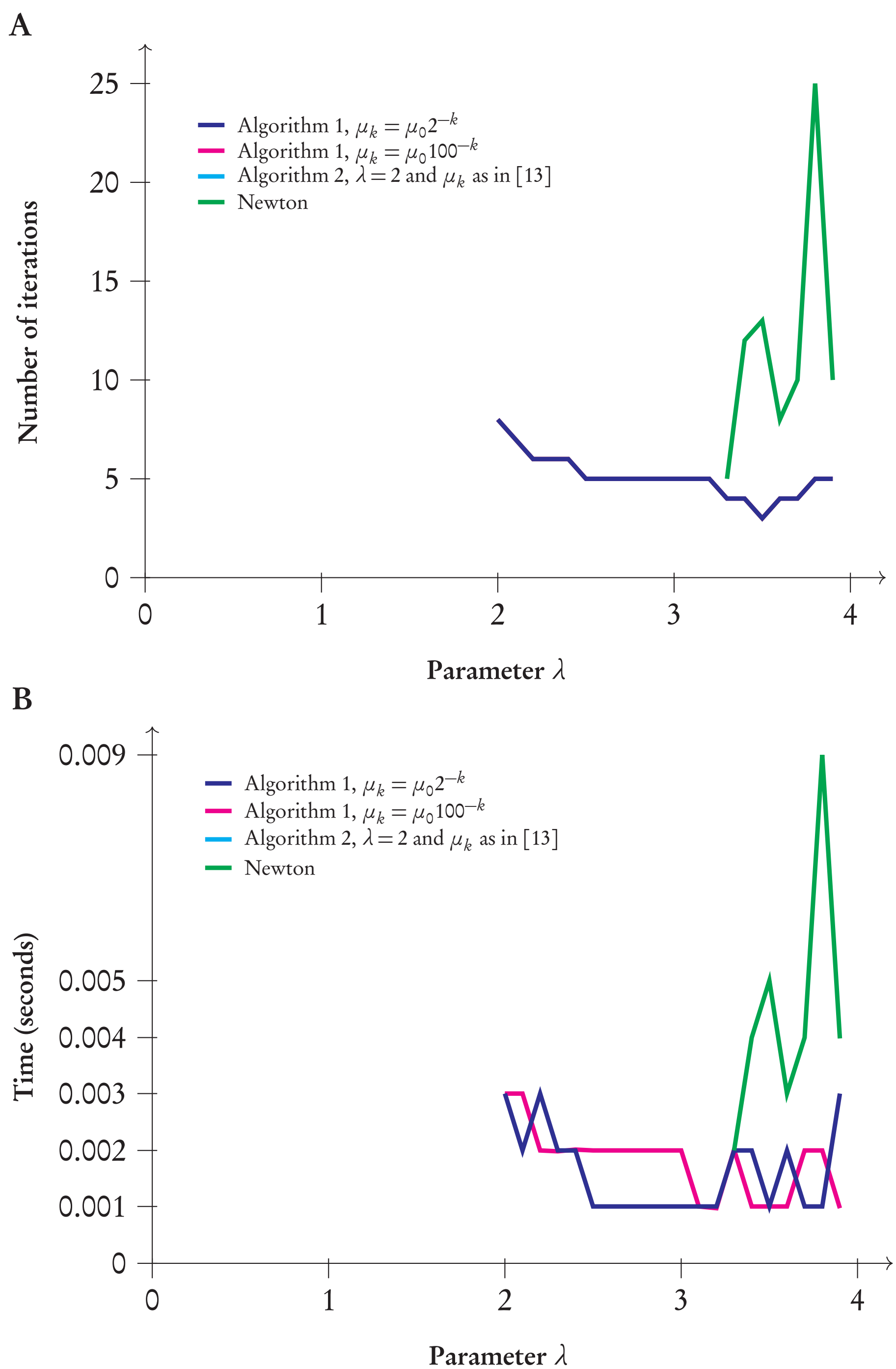

Figure 1. For the *Billups* function, the Algorithm 2 has a better performance in both number of iterations (**A**) and computational time (**B**). For $\lambda \in [0, 2)$ there is no graph because Algorithms 1 and 2 diverge.

Fig. 2 shows that convergence was obtained for all the algorithms and all the values of λ. The number of iterations for the *Newton* method is slightly smaller than the others ones (Fig. 2A); the proposed smoothing has a fairly competitive behavior. In terms of computational time, the proposed algorithm lies between the generalized *Newton* method and the Algorithm 2 (Fig. 2B).

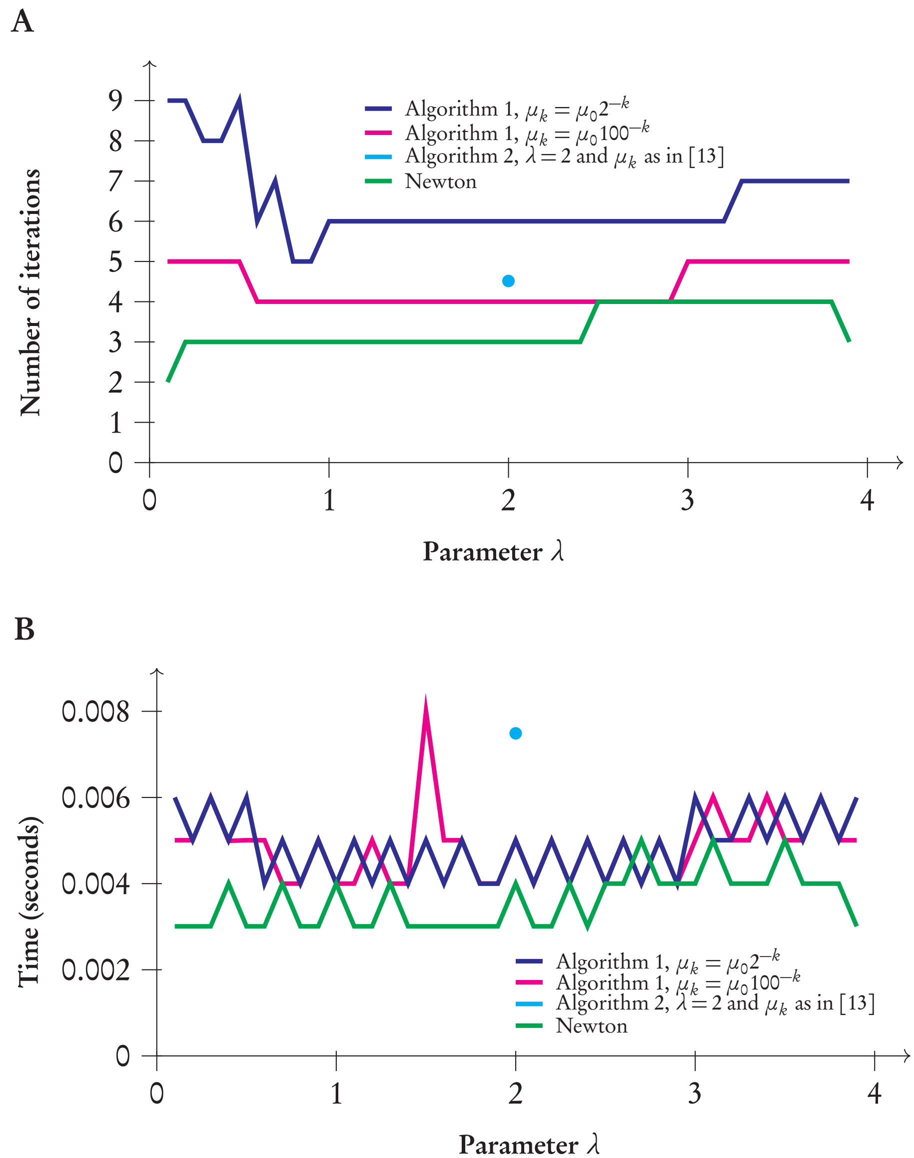

Figure 2. For the *Mathiesen* function, the algorithms are competitive in number of iterations (**A**) and computational time (**B**).

In **Fig.3**, we observe that the generalized *Newton* method converges for $\lambda \geq 1.8$, and the proposed algorithm converges for $\lambda \geq 1.5$; in terms of number of iterations and convergence time, these two methods have a similar behavior for $1.5 \leq \lambda \leq 3.1$. For $\lambda > 3.1$ the proposed algorithm is the best.

Fig.4 shows that the generalized *Newton* method and the proposed algorithm converge for $\lambda \geq 2$; the number of iterations is the same for the two sequences $\{\mu_k\}$ used and for the generalized *Newton* method; In addition, the Algorithm 2 converges using more iterations and spending more computational time than the other methods.

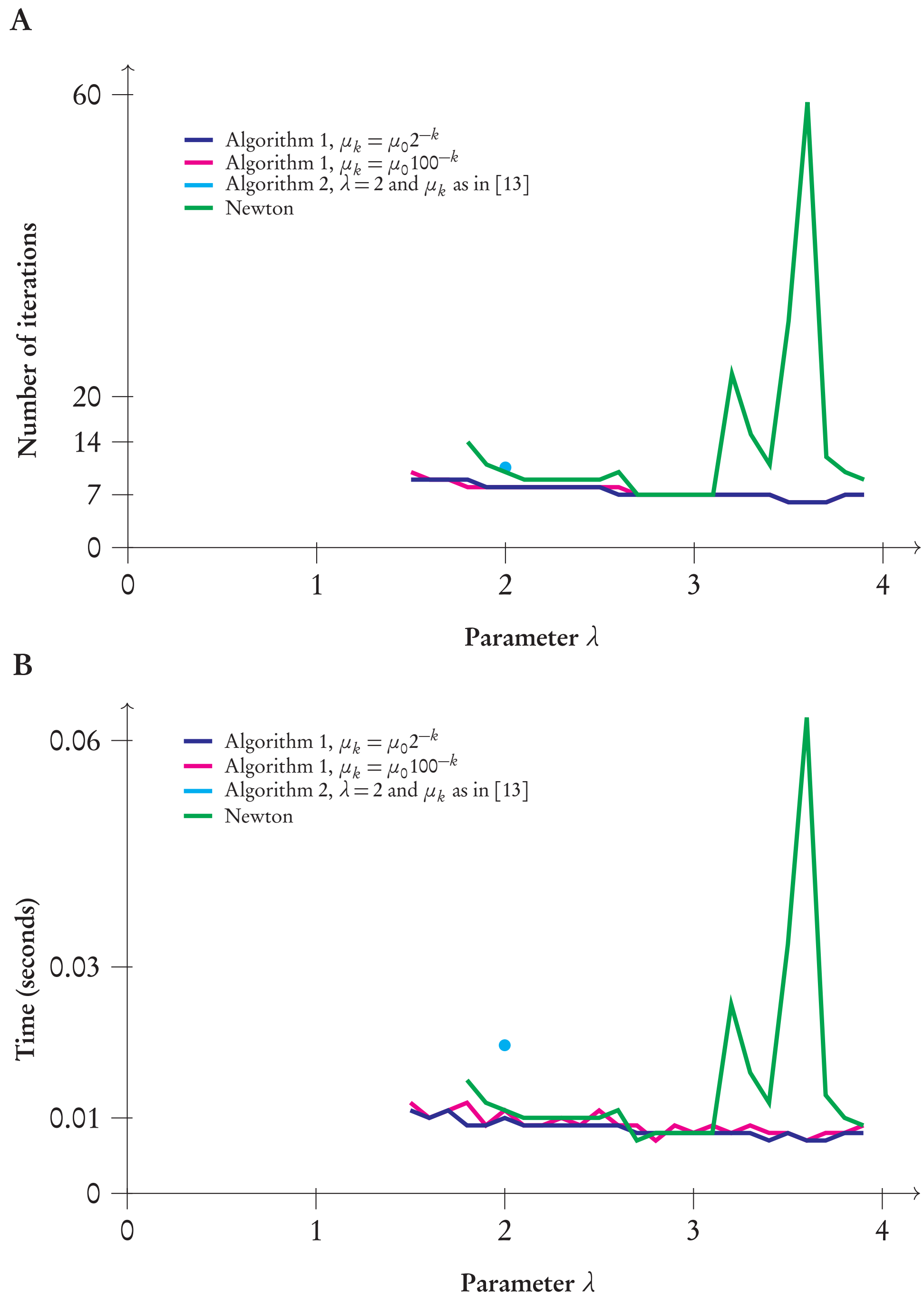

Figure 3. For the *Kojima-Shindo* function, the algorithm proposed is better than the other ones from $\lambda > 3.1$ in both number of iterations (**A**) and computational time (**B**). For $\lambda \in [0, 1.8)$ there are no graphs because Algorithms 1 and 2 diverge.

Finally, in **Fig.5**, we observe convergence of all the considered methods, for all values of λ. Again, it is shown that the proposed algorithm, Algorithm 1, is quite competitive with respect to the *Newton* method. On the other hand, in our proposal, the selection of the sequence μ_k has an important role in the convergence rate.

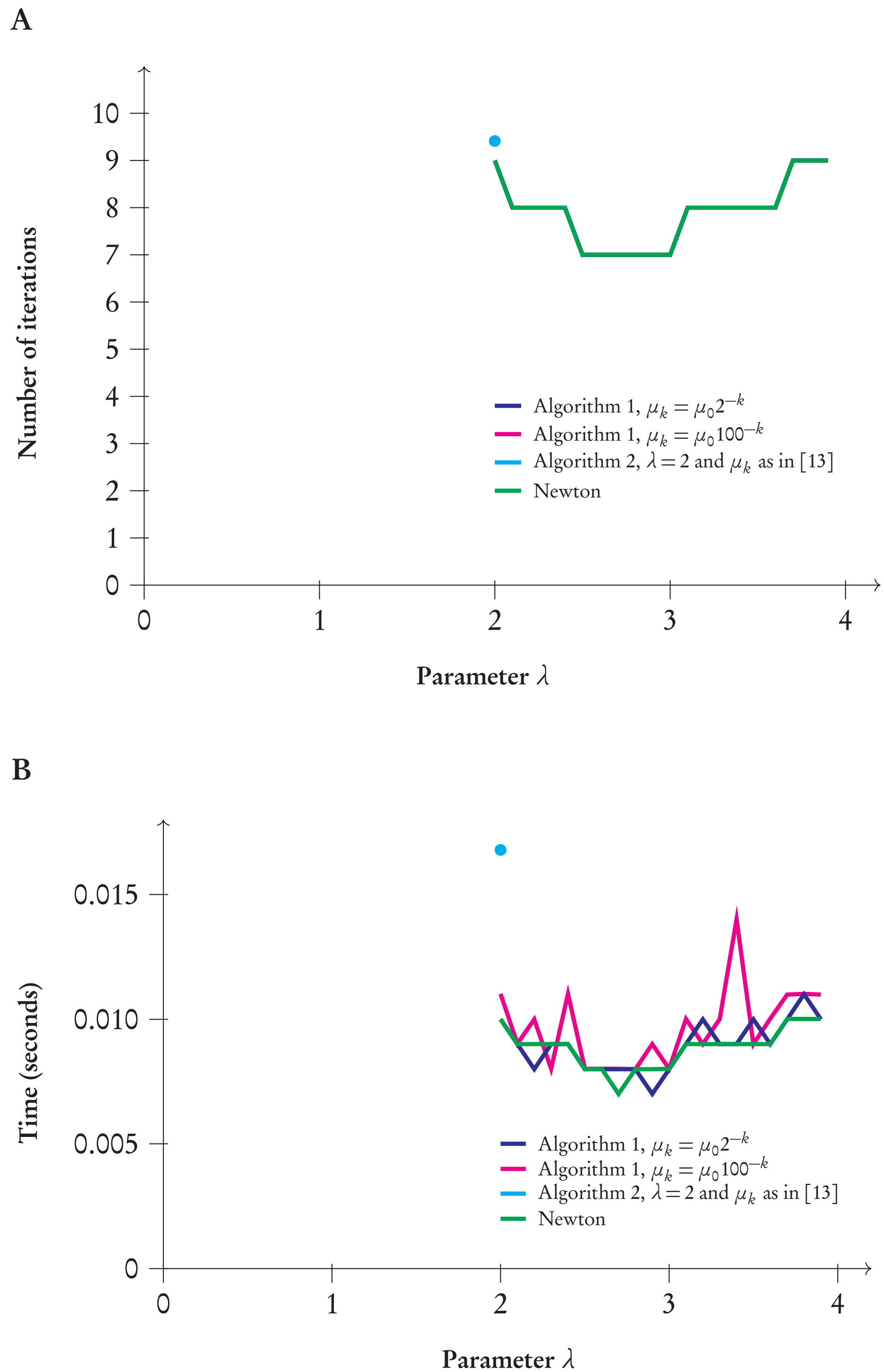

Figure 4. For *Kojima–Josephy* function, Algorithm 1 (with two sequences used) and the *Newton* method have the same number of iterations, for this reason their graphics coincide (**A**). The Algorithm 2 is slower computationally (**B**). In addition, for $\lambda \in [0, 2)$ there is no graph because Algorithms 1 and 2 diverge.

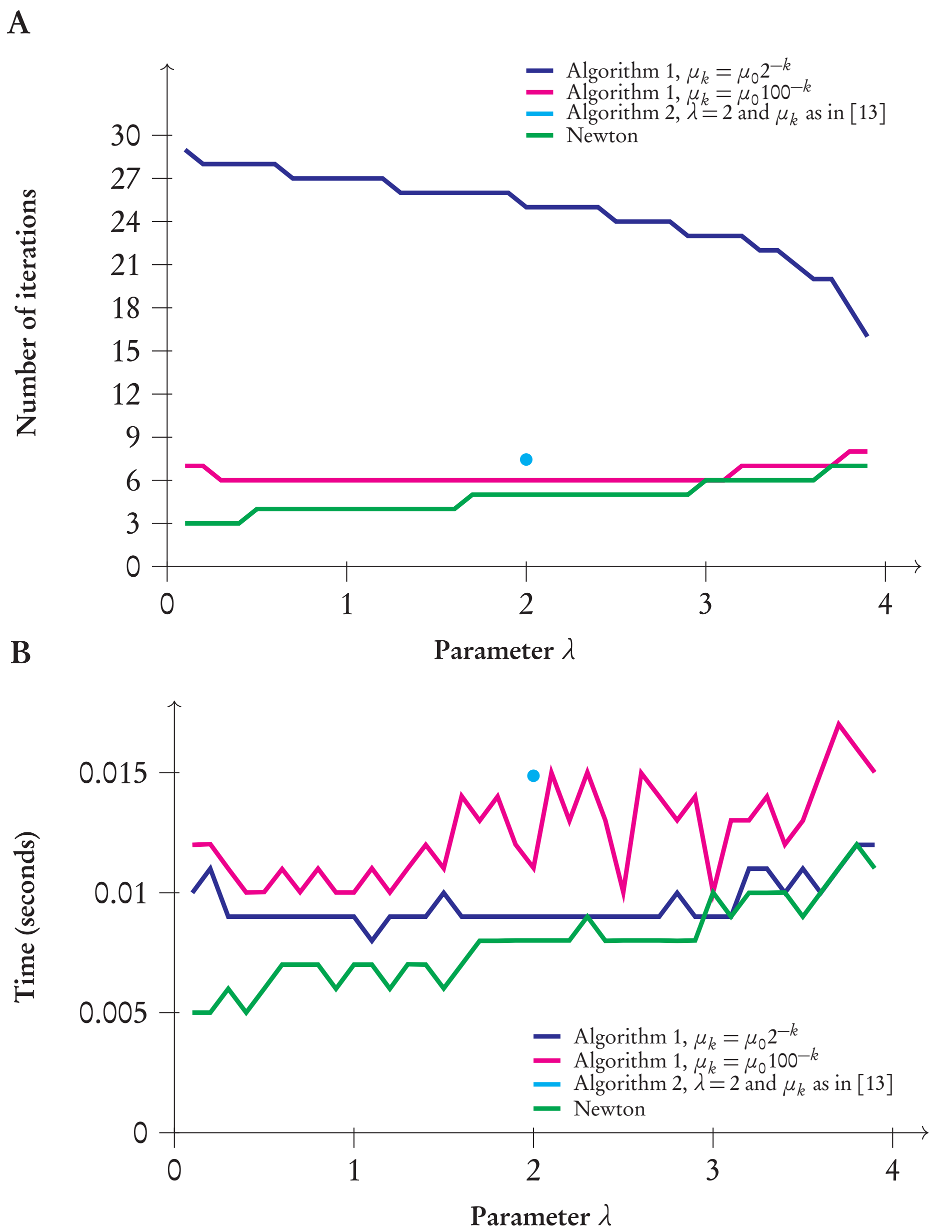

Figure 5. NCP with the function given by Example A in [24] shows another case in which the proposed algorithm is better than the other algorithms in number of iterations and computational time.

Concluding remarks

In this paper, we propose a new generalized Newton-type algorithm for solving Nonlinear Complementarity Problems based on its reformulation as a nonsmooth system of equations. To do this, we introduce a smoothing of the family of nonlinear complementarity functions presented in [7] and analyze its properties in combination with the smooth Jacobian strategy used in [17]. We prove local and quadratic convergence for the new algorithm.

We present preliminary numerical tests that show a competitive numerical performance of the proposed algorithm compared to the traditional generalized *Newton* method and the Jacobian smoothing *Newton* method that uses the *Fischer–Burmeister* complementarity function, which is a particular case of the family that we consider in our proposal. These numerical tests allow us to observe that the choice of the sequence μ_k plays an important role in the convergence of the algorithm.

We consider that more numerical tests are needed varying the choice of λ as well as $\{\mu_k\}$. Moreover, the globalization of the proposed algorithm with its global convergence analysis is also needed.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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Un método local de suavización Jacobiana para resolver Problemas de Complementariedad No Lineal

Resumen: En este artículo, presentamos un método de suavización para una familia de funciones de complementariedad no lineales y utilizamos sus propiedades, en combinación con la estrategia Jacobiana para el caso suave, con el propósito de introducir un nuevo algoritmo generalizado de tipo Newton para resolver un sistema no suave de ecuaciones equivalente al Problema de Complementariedad No Lineal. Además, demostramos que el algoritmo converge localmente y q-cuadráticamente, y analizamos su rendimiento numérico.

Palabras clave: problema de complementariedad no lineal; función de complementariedad; método de Newton generalizado; convergencia Q-cuadrática.

Um método local de regularização Jacobiana para resolver Problemas de Complementaridade Não-lineares

Resumo: Neste artigo, apresentamos um método de suavização para uma família de funções de complementaridade não lineares e utilizamos suas propriedades, em combinação com a estratégia Jacobiana para o caso suave, a fim de introduzir um novo algoritmo generalizado do tipo Newton para resolver um sistema não suave de equações equivalentes ao Problema de Complementaridade Não Linear. Além disso, demonstramos que o algoritmo converge localmente e *q*-quadraticamente, e analisamos seu desempenho numérico.

Palavras-chave: problema de complementaridade não linear; função de complementaridade; método de Newton generalizado; convergência *Q*-quadrática.

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