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Product of deferred Cesàro and deferred weighted statistical probability convergence and its applications to Korovkin-type theorems

Bidu Bhusan Jena[1], Susanta Kumar Paikray[1,*]

Edited by
Juan Carlos Salcedo-Reyes
(salcedo.juan@javeriana.edu.co)

1. Veer Surendra Sai University of Technology.

* skpaikray_math@vssut.ac.in

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Abstract In the present work, we introduce and study the notion of statistical probability convergence for sequences of random variables as well as the idea of statistical convergence for sequences of real numbers, which are defined over a Banach space via the product of deferred Cesàro and deferred weighted summability means. We first establish a theorem presenting aconnection between them. Based upon our proposed methods, we then prove a Korovkin-type approximation theorem with algebraic test functions for a sequence of random variables on a Banach space, and demonstrate that our theorem effectively extends and improves most (if not all) of the previously existing results (in classical as well as in statistical versions). Furthermore, an illustrative example is presented here by means of the generalized Meyer–König and Zeller operators of a sequence of random variables in order to demonstrate that our established theorem is stronger than its traditional and statistical versions. Finally, we estimate the rate of the product of deferred Cesàro and deferred weighted statistical probability convergence, and accordingly establish a new result.

Keywords: Statistical convergence; Statistical probability convergence; Deferred Cesàro and Deferred weighted product mean; Positive linear op-erators; Sequence of random variables; Banach space; Korovkin-type theo-rems; Rate of statistical probability convergence.

Introduction and Motivation

The gradual evolution on convergence of sequence spaces lead to the development of a beautiful concept known as “statistical convergence" and such concept was first introduced independently by two eminent mathematicians Fast [1] and Steinhaus [2]. It is more important than the usual convergence because the traditional convergence of a sequence requires that almost all elements are to satisfy the convergence condition, that is, every element of the sequence needs to be in some neighborhood (arbitrarily small) of the limit. However, such restriction is relaxed in statistical convergence, where set having a few elements that are not in the neighborhood of the limit is discarded subject to the condition that the natural density of the set is zero, and at the same time the condition of convergence is valid for the other majority of the elements. Actually, a root of the notion of statistical convergence was discussed by Zygmund (see [3, p. 181]), where he used the term "almost convergence", which turned out to be equivalent to the concept of statistical convergence. We also find such concepts in random graph theory (see [4, 5]) in the sense that almost convergence means convergence with probability 1, whereas in statistical convergence the probability is not necessarily 1. Mathematically, a sequence of random variables $\{X_n\}$ is statistically probability convergent (converges in probability) to a random variable X if $\lim_{n\to\infty} P(|X_n - X| \geqq \epsilon) = 0$, for all $\epsilon > 0$ (arbitrarily small); and almost convergent to X if $P(\lim_{n\to\infty} X_n = X) = 1$. Recently, this hypothesis is analyzed in the various fields of pure and applied mathematics such as real analysis, Fourier analysis, measure theory, probability theory, and approximation theory. For current works see [6–15].

Let $\mathbb{N}$ be the set of natural numbers and let $K \subseteq \mathbb{N}$. Also, let

$$K_n = \{k\colon k \leqq n \quad \text{and} \quad k \in K\}$$

and suppose that $|K_n|$ is the cardinality of K_n. Then the natural density $d(K)$ of K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k\colon k \leqq n \quad \text{and} \quad k \in K\}|,$$

provided that the limit exists.

A given real sequence (x_n) is said to be statistically convergent to L if, for each $\epsilon > 0$, the set

$$K_\epsilon = \{k\colon k \in \mathbb{N} \quad \text{and} \quad |x_k - L| \geqq \epsilon\}$$

has zero natural density (see [1] and [2]). Thus, for each $\epsilon > 0$, we have

$$d(K_\epsilon) = \lim_{n\to\infty} \frac{|K_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k\colon k \leqq n \quad \text{and} \quad |x_k - L| \geqq \epsilon\}| = 0.$$

In this case, we write

$$\operatorname{stat}\lim_{n\to\infty} x_n = L.$$

In the year 2002, Móricz [16] initially presented the elementary concept of statistical Cesàro summability. Later on, Mohiuddine *et al.* [17] established some approximation theorems of the Korovkin-type based upon the statistical Cesàro summability. Subsequently, Karakaya and Chishti [18] introduced and studied the concept of weighted statistical convergence and their definition was later modified by Mursaleen *et al.* [19]. Furthermore, the fundamental concept of the deferred Cesàro statistical convergence as well as of the statistically-deferred Cesàro summability and associated approximation theorems was introduced by Jena *et al.* [20]. Recently, Srivastava *et al.* [21] introduced the notion of deferred weighted statistical convergence and proved analogous approximation theorems and also, in the same year Srivastava *et al.* [22] proved equi-statistical convergence via deferred Nörlund summability mean and accordingly established new approximation of the Korovkin-type theorems. Very recently, Jena *et al.* [6] studied product of statistical convergence in probability and established analogous Korovkin-type approximation theorems for algebraic test functions. Subsequently, Jena *et al.* [23] also introduced various fundamental limit concept of statistical probability convergence and accordingly proved some approximation theorems via deferred Cesàro summability means. For several other recent developments in this direction, see for example [21, 22, 24–28].

Recalling the probability theory, let $X_n \quad (n \in \mathbb{N})$ be a random variable defined on an event space S with respect to a given class of events Δ. Let $P\colon \Delta \to \mathbb{R}$ (where $\mathbb{R}$ is the set of real numbers) be a probability density function. Then we denote the sequence $X_1, X_2, X_3, \ldots$ of random variables by $\{X_n\}_{n\in\mathbb{N}}$.

Moreover, this study will be interesting, if there exists a constant $c \in \mathbb{R}$ such that for given $\epsilon > 0$ (arbitrarily small) for which

$$P(|X - c| < \epsilon) = 1.$$

Furthermore, for a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$, where every X_n may not satisfy the above property; however it may be possible that, for sufficiently large n the above property becomes more significant corresponding to a constant $c \in \mathbb{R}$. Next, the existence of such c will be addressed by the notion of probability convergence (that is, convergence in probability) for the sequence $\{X_n\}_{n\in\mathbb{N}}$.

Suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of random variables, and each X_n is defined over the event space (same) S, corresponding to a given class of events Δ as the subsets (of S) under the probability function $P\colon \Delta \to \mathbb{R}$. The sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be statistically probability convergent (or statistically convergent in probability) to a random variable X (where $X\colon S \to \mathbb{R}$) if, for any $\epsilon > 0$ and $\delta > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |k\colon k \leqq n \quad \text{and} \quad P(|X_n - X| \geqq \epsilon) \geqq \delta| = 0$$

or, equivalently,

$$\lim_{n\to\infty} \frac{1}{n} |k\colon k \leqq n \quad \text{and} \quad 1 - P(|X_n - X| \leqq \epsilon) \geqq \delta| = 0.$$

In this case, we write

$$\mathrm{stat_P} \lim_{n\to\infty} P(|X_n - X| \geqq \epsilon) = 0 \qquad \text{or} \qquad \mathrm{stat_P} \lim_{n\to\infty} P(|X_n - X| \leqq \epsilon) = 1.$$

We now show by means of the following example that every statistically convergent sequence is statistically probability convergent, but the converse is not necessarily true.

Example 1.1. Consider a probability density function of X_n given for $n = m^2$ and for all $m \in \mathbb{N}$ by

$$f_n(x) = \begin{cases} \frac{1}{3} & (0 < x < 3) \\ 0 & (\text{otherwise}) \end{cases}$$

and for $n \neq m^2$, for all $m \in \mathbb{N}$,

$$f_n(x) = \begin{cases} \dfrac{nx^{n-1}}{5^n} & (0 < x < 5) \\ 0 & (\text{otherwise}). \end{cases}$$

Let $0 < \epsilon, \delta < 1$. Then

$$P(|X_n - 5| \geqq \epsilon) = \begin{cases} \frac{1}{3} & (n = m^2, \text{ for all } m \in \mathbb{N}) \\ 1 - P(|X_n - 5| < \epsilon) & \\ \quad = \left(1 - \dfrac{\epsilon}{5}\right)^n & (n \neq m^2, \text{ for all } m \in \mathbb{N}). \end{cases}$$

This implies that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} |\{k\colon k \leqq n \quad \text{and} \quad P(|X_n - 5| \geqq \epsilon) \geqq \delta\}| \\ &\qquad \leqq \lim_{n\to\infty} \frac{1}{n} |\{1^2, 2^2, 3^2, \cdots, n^2\}| = 0. \end{aligned}$$

Clearly, it is neither statistically convergent nor ordinarily convergent, while it is statistically probability convergent to 5.

Quite recently, Srivastava *et al.* [29] first introduced and studied the fundamental idea of deferred Cesàro statistical probability convergence of a sequence of random variables as follows.

A given sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be deferred Cesàro statistically probability convergent to a random variable X (where $X\colon S\to\mathbb{R}$), if for every $\delta>0$ and $\epsilon>0$, the set

$$\{k\colon a_n<k\leqq b_n \quad \text{and} \quad P(|X_n-X|\geqq\epsilon)\geqq\delta\}$$

has natural density zero, that is,

$$\lim_{n\to\infty}\frac{1}{b_n-a_n}|\{k\colon a_n<k\leqq b_n \quad \text{and} \quad P(|X_n-X|\geqq\epsilon)\geqq\delta\}|=0.$$

In this case, we write

$$\mathrm{stat}_{\mathrm{DCP}}\lim_{n\to\infty}P(|X_n-X|\geqq\epsilon)=0.$$

Several researchers have worked on extending or generalizing the approximation theorems of the Korovkin-type in many different ways and under various different settings, together with Banach spaces, Banach algebras and function spaces etc. In the year 2018, Jena *et al.* [20] introduced statistically-deferred Cesàro summability for single sequences in Korovkin-type approximation theorems. Recently, Paikray *et al.* [30] established a Korovkin-type theorem based upon the (p,q)-integers for statistically-deferred Cesàro summability mean. Subsequently, Dutta *et al.* [31] demonstrated the Korovkin theorem on $\mathcal{C}[0,\infty)$ by using the test functions 1, e^{-x} and e^{-2x} via the deferred Cesàro mean. In another recent work, Srivastava *et al.* [21] made use of the notion of the deferred weighted statistical convergence and accordingly proved a Korovkin-type approximation theorem.

Motivated essentially by the above-mentioned investigations and results, we first introduced here the concept of the product of deferred Cesàro and deferred weighted statistical convergence of real sequences, and then for the statistical probability convergence of sequences with random variables. We also established an inclusion relation between them. Moreover, based upon our proposed methods, we proved a new Korovkin-type approximation theorem with algebraic test functions for positive sequences of random variables over a Banach space and demonstrated that our result is a non-trivial extension of some well-established traditional and statistical versions of several known results. Finally, we estimated the rate of the product of deferred Cesàro and deferred weighted statistical probability convergence and accordingly established a new result.

Preliminaries and Definitions

Let (a_n) and (b_n) be sequences of non-negative integers such that, (i) $\lim_{n\to\infty} b_n = \infty$ and (ii) $a_n < b_n$, then the deferred Cesàro (DC) mean is given by (see, Agnew [32, p. 414]),

$$\begin{aligned}\sigma_n &= \frac{x_{a_n+1} + x_{a_n+2} + x_{a_n+3} + \ldots + x_{b_n}}{b_n - a_n} \\ &= \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} x_k.\end{aligned}$$

It is trivial that, under the above conditions (i) and (ii), DC mean is regular (also, see Agnew [32]).

Similarly, suppose that (p_n) is a sequence of non-negative real numbers such that

$$\mathcal{P}_n = \sum_{m=a_n+1}^{b_n} p_m,$$

then the deferred weighted (DW) mean is defined by (see [22])

$$\theta_n = \frac{1}{\mathcal{P}_n} \sum_{m=a_n+1}^{b_n} p_m x_m.$$

It is well known that, DW mean is regular under the above-mentioned conditions (i) and (ii) (see, for details, Agnew [32]).

We now define the product [D(CW)] of DC and DW means as follows:

$$\begin{aligned}\Omega_n = (\sigma\theta)_n &= \frac{1}{(b_n - a_n)} \sum_{m=a_n+1}^{b_n} (\theta_m) \\ &= \frac{1}{(b_n - a_n)} \sum_{m=a_n+1}^{b_n} \frac{1}{\mathcal{P}_m} \sum_{v=a_m+1}^{b_m} p_v x_v.\end{aligned}$$

Moreover, the sequence (Ω_n) is summable to L by the product [D(CW)] summability mean if,

$$\lim_{n\to\infty} \Omega_n = L.$$

Also, we assume that the [D(CW)] product mean is regular.

Let us now introduce the following definitions which will be needed in connection with our proposed investigation here.

Definition 2.1. Let (a_n) and (b_n) be sequences of non-negative integers and let (p_n) be the sequence of non-negative real numbers. A real sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be deferred Cesàro and deferred weighted statistically product convergent (or $[D(CW)]_s$ convergent) to L if, for each $\epsilon > 0$, the set given by

$$\{m\colon m \leqq (b_n - a_n)\mathcal{P}_n \quad \text{and} \quad p_m|x_m - L| \geqq \epsilon\}$$

has its natural density equal to zero, that is, if

$$\lim_{n\to\infty} \frac{1}{(b_n - a_n)\mathcal{P}_n} |\{m\colon m \leqq (b_n - a_n)\mathcal{P}_n \quad \text{and} \quad p_m|x_m - L| \geqq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat}_{[\mathrm{D(CW)}]} \lim_{n\to\infty} x_n = L \quad \text{or} \quad [D(CW)]_s \lim_{n\to\infty} x_n = L.$$

Definition 2.2. Let (a_n) and (b_n) be sequences of non-negative integers and let (p_n) be the sequence of non-negative real numbers. Suppose also that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of random variables, where each (X_n) is defined on the same event space S with respect to a given class Δ of subsets of the event space S and a given probability density function $P\colon \Delta \to \mathbb{R}$. A given sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be deferred Cesàro and deferred weighted statistically probability convergent (or $[D(CW)]_{sp}$ convergent) to a random variable X (where $X\colon S \to \mathbb{R}$) if, for every $\delta > 0$ and $\epsilon > 0$, the set

$$\{m\colon m \leqq (b_n - a_n)\mathcal{P}_n \quad \text{and} \quad p_m P(|X_n - X| \geqq \epsilon) \geqq \delta\}$$

has its natural density equal to zero, that is, if

$$\lim_{n\to\infty} \frac{1}{(b_n - a_n)\mathcal{P}_n} |\{m\colon m \leqq (b_n - a_n)\mathcal{P}_n,\ p_m P(|X_n - X| \geqq \epsilon) \geqq \delta\}| = 0$$

or, equivalently,

$$\lim_{n\to\infty} \frac{1}{(b_n - a_n)\mathcal{P}_n} |\{m\colon m \leqq (b_n - a_n)\mathcal{P}_n,\ 1 - p_m P(|X_n - X| \leqq \epsilon) \geqq \delta\}| = 0.$$

In this case, we write

$$\text{stat}_{[\mathrm{D(CW)P}]} \lim_{n\to\infty} p_m P(|X_n - X| \geqq \epsilon) = 0 \quad \text{or}$$
$$[D(CW)]_{sp} \lim_{n\to\infty} p_m P(|X_n - X| \geqq \epsilon) = 0$$

or, equivalently, we also write

$$\text{stat}_{[\mathrm{D(CW)P}]} \lim_{n\to\infty} p_m P(|X_n - X| \leqq \epsilon) = 1 \quad \text{or}$$
$$[D(CW)]_{sp} \lim_{n\to\infty} p_m P(|X_n - X| \leqq \epsilon) = 1.$$

We next present a theorem in order to demonstrate that every $[D(CW)]_s$ convergent sequence is $[D(CW)]_{sp}$ convergent. However, the converse is not true.

Theorem 2.3. *Let the sequence $\{x_n\}$ of constants be such that $\text{stat}_{[\text{D(CW)}]}x_n \to x$. Then, assuming it to be a random variable having a one-point distribution at that point, the sequence $\{X_n\}$ of random variables is such that*

$$\text{stat}_{[\text{D(CW)P}]}X_n \to X.$$

Proof. Let $\epsilon > 0$ be any arbitrarily small positive real number. Then, by Definition 2.1, we obtain

$$\lim_{n\to\infty} \frac{1}{(b_n - a_n)\mathcal{P}_n} |\{m \colon m \leqq (b_n - a_n)\mathcal{P}_n \quad \text{and} \quad p_m|x_m - L| \geqq \epsilon\}| = 0.$$

We now let $\delta > 0$, so that the set

$$\{m \colon m \leqq (b_n - a_n)\mathcal{P}_n \quad \text{and} \quad p_m P(|X_n - X| \geqq \epsilon) \geqq \delta\} \subseteq \mathcal{K},$$

where

$$\mathcal{K} = \{m \colon m \leqq (b_n - a_n)\mathcal{P}_n \quad \text{and} \quad p_m|x_m - L| \geqq \epsilon\}.$$

Thus, by Definition 2.2, we may write

$$\text{stat}_{[\text{D(CW)P}]}\, X_n \to X. \qquad \square$$

We now present below an example to show that a sequence of random variables is $[D(CW)]_{sp}$ convergent, whenever it is not $[D(CW)]_s$ convergent.

Example 2.4. Let $a_n = 2n - 1$, $b_n = 4n - 1$ and $p_n = n$. Suppose that the probability density functions of X_n is given by $n = m^2$, for all $m \in \mathbb{N}$,

$$f_n(x) = \begin{cases} \frac{1}{2} & (0 < x < 2) \\ 0 & (\text{otherwise}) \end{cases}$$

and $n \neq m^2$, for all $m \in \mathbb{N}$,

$$f_n(x) = \begin{cases} \frac{(n+1)x^n}{5^{n+1}} & (0 < x < 5) \\ 0 & (\text{otherwise}). \end{cases}$$

Let $0 < \epsilon, \delta < 1$. Then

$$P(|X_n - 5| \geqq \epsilon) = \begin{cases} \frac{1}{2} & \text{when} \quad n = m^2 \\ 1 - P(|X_n - 5| < \epsilon) = \left(1 - \frac{\epsilon}{5}\right)^n & \text{when} \quad n \neq m^2. \end{cases}$$

Consequently, we have

$$\lim_{n\to\infty} \frac{1}{2n^2} |\{m \colon m \leqq 2n^2 \quad \text{and} \quad nP(|X_n - 5| \geqq \epsilon) \geqq \delta\}| = 0.$$

Clearly, we observe that (X_n) is neither convergent nor $[D(CW)]_s$ convergent; however, it is $[D(CW)]_{sp}$ convergent to 5.

A New Korovkin-type Theorem

In this section, we extend here the result of Jena *et al.* [20] and Srivastava *et al.* [21] by using the product deferred Cesàro and deferred weighted convergence (that is, $[D(CW)]_{sp}$ convergence) of sequences of random variables over a Banach space.

Let $\mathcal{C}(X)$ be the continuous real valued probability functions defined over a compact set X ($X \subset \mathbb{R}$) with the supremum norm $\|.\|_\infty$. Also let $\mathcal{C}(X)$ be a Banach space. Then, for each $f \in \mathcal{C}(X)$, the norm of f (denoted by $\|f\|_\infty$) is given by,

$$\|f\|_\infty = \sup_{x \in X}\{|f(x)|\}.$$

We say that the operator $\mathfrak{L}$ is a sequence of random variables of positive linear operator provided that

$$\mathfrak{L}(f;x) \geqq 0 \quad \text{whenever} \quad f \geqq 0.$$

Now we prove the following theorem by using the product deferred Cesàro and deferred weighted $[D(CW)]_{sp}$ convergence.

Theorem 3.1. *Let*

$$\mathfrak{L}_m \colon \mathcal{C}(X) \to \mathcal{C}(X)$$

be a sequence of random variables of positive linear operators. Then, for all $f \in \mathcal{C}(X)$,

$$\text{stat}_{[\text{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0, \tag{1}$$

if and only if

$$\text{stat}_{[\text{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0, \tag{2}$$

$$\text{stat}_{[\text{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(x;x) - x\|_\infty = 0,$$

and

$$\text{stat}_{[\text{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(x^2;x) - x^2\|_\infty = 0. \tag{3}$$

Proof. Since each of the following functions

$$f_0(x) = 1, \quad f_1(x) = x, \quad \text{and} \quad f_2(x) = x^2$$

belonging to $\mathcal{C}(X)$ are continuous, the implication given by (1) implies (2) to (3) is fairly obvious. Next, for the completion of the proof of the Theorem 3.1, we first assume that the conditions (2) to (3) hold true. If $f \in \mathcal{C}(X)$, then there exists a constant $\mathcal{M} > 0$ such that

$$|f(x)| \leqq \mathcal{M} \quad (\forall\ x \in X).$$

We thus find that

$$|f(s) - f(x)| \leqq 2\mathcal{M} \quad (s, x \in I).$$

Clearly, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(s) - f(x)| < \epsilon \tag{4}$$

whenever

$$|s - x| < \delta, \quad \text{for all } s, x \in I.$$

Let us choose

$$\varphi_1 = \varphi_1(s, x) = (s - x)^2.$$

If $|s - x| \geqq \delta$, then we obtain

$$|f(s) - f(x)| < \frac{2\mathcal{M}}{\delta^2}\varphi_1(s, x). \tag{5}$$

From equation (4) and (5), we get

$$|f(s) - f(x)| < \epsilon + \frac{2\mathcal{M}}{\delta^2}\varphi_1(s, x),$$

which implies that

$$-\epsilon - \frac{2\mathcal{M}}{\delta^2}\varphi_1(s, x) \leqq f(s) - f(x) \leqq \epsilon + \frac{2\mathcal{M}}{\delta^2}\varphi_1(s, x).$$

Now since $\mathfrak{L}_m(1; x)$ is monotone and linear, applying the operator $\mathfrak{L}_m(1; x)$ to this inequality, we have

$$\begin{aligned}\mathfrak{L}_m(1; x)\left(-\epsilon - \frac{2\mathcal{M}}{\delta^2}\varphi_1(s, x)\right) &\leqq \mathfrak{L}_m(1; x)(f(s) - f(x)) \\ &\leqq \mathfrak{L}_m(1; x)\left(\epsilon + \frac{2\mathcal{M}}{\delta^2}\varphi_1(s, x)\right).\end{aligned}$$

We note that x is fixed and so $f(x)$ is a constant number. Therefore, we have

$$\begin{aligned}-\epsilon\mathfrak{L}_m(1; x) - \frac{2\mathcal{M}}{\delta^2}\mathfrak{L}_m(\varphi_1; x) &\leqq \mathfrak{L}_m(f; x) - f(x)\mathfrak{L}_m(1; x) \\ &\leqq \epsilon\, \mathfrak{L}_m(1; x) + \frac{2\mathcal{M}}{\delta^2}\mathfrak{L}_m(\varphi_1; x).\end{aligned} \tag{6}$$

But

$$\mathfrak{L}_m(f; x) - f(x) = [\mathfrak{L}_m(f; x) - f(x)\mathfrak{L}_m(1; x)] + f(x)[\mathfrak{L}_m(1; x) - 1]. \tag{7}$$

Using (6) and (7), we have

$$\mathfrak{L}_m(f;x) - f(x) < \epsilon\mathfrak{L}_m(1;x) + \frac{2\mathcal{M}}{\delta^2}\mathfrak{L}_m(\varphi_1;x) + f(x)[\mathfrak{L}_m(1;x) - 1]. \tag{8}$$

We now estimate $\mathfrak{L}_m(\varphi_1;x)$ as follows:

$$\begin{aligned}\mathfrak{L}_m(\varphi_1;x) &= \mathfrak{L}_m((s-x)^2;x) = \mathfrak{L}_m(s^2 - 2xs + x^2;x)\\ &= \mathfrak{L}_m(s^2;x) - 2x\mathfrak{L}_m(s;x) + x^2\mathfrak{L}_m(1;x)\\ &= [\mathfrak{L}_m(s^2;x) - x^2] - 2x[\mathfrak{L}_m(s;x) - x]\\ &\quad + x^2[\mathfrak{L}_m(1;x) - 1].\end{aligned}$$

Using (8), we obtain

$$\begin{aligned}\mathfrak{L}_m(f;x) - f(x) &< \epsilon\,\mathfrak{L}_m(1;x) + \frac{2\mathcal{M}}{\delta^2}\{[\mathfrak{L}_m(s^2;x) - x^2]\\ &\qquad - 2x[\mathfrak{L}_m(s;x) - e^{-x}] + x^2[\mathfrak{L}_m(1;x) - 1]\}\\ &\qquad + f(x)[\mathfrak{L}_m(1;x) - 1].\\ &= \epsilon\,[\mathfrak{L}_m(1;x) - 1] + \epsilon + \frac{2\mathcal{M}}{\delta^2}\{[\mathfrak{L}_m(s^2;x) - x^2]\\ &\qquad - 2x[\mathfrak{L}_m(s;x) - x] + x^2[\mathfrak{L}_m(1;x) - 1]\}\\ &\qquad + f(x)[L_m(1;x) - 1].\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we can write

$$\begin{aligned}|\mathfrak{L}_m(f;x) - f(x)| &\leqq \epsilon + \left(\epsilon + \frac{2\mathcal{M}}{\delta^2} + \mathcal{M}\right)|\mathfrak{L}_m(1;x) - 1|\\ &\quad + \frac{4\mathcal{M}}{\delta^2}|\mathfrak{L}_m(s;x) - x| + \frac{2\mathcal{M}}{\delta^2}|\mathfrak{L}_m(s^2;x) - x^2|\\ &\leqq \mathcal{K}(|\mathfrak{L}_m(1;x) - 1| + |\mathfrak{L}_m(s;x) - x|\\ &\qquad + |\mathfrak{L}_m(s^2;x) - x^2|),\end{aligned}$$

where

$$\mathcal{K} = \max\left(\epsilon + \frac{2\mathcal{M}}{\delta^2} + \mathcal{M}, \frac{4\mathcal{M}}{\delta^2}, \frac{2\mathcal{M}}{\delta^2}\right).$$

Now, for a given $r > 0$, there exists $\delta, \epsilon > 0$, such that $\epsilon < r$. Then, by setting

$$\Omega_m(x;r) = \{m\colon m \leqq (b_n - a_n)\mathcal{P}_n;\ p_mP\left(|\mathfrak{L}_m(f;x) - f(x)| \geqq r\right)\} \geqq \delta.$$

Also, for $i = 0, 1, 2$, the set $\Omega_{i,m}(x;r)$ equals

$$\left\{m\colon m \leqq (b_n - a_n)\mathcal{P}_n;\ p_mP\ \ \left(|\mathfrak{L}_m(f_i;x) - f_i(x)| \geqq \frac{r-\epsilon}{3\mathcal{K}}\right) \geqq \delta\right\},$$

so that,

$$\Omega_m(x;r) \leqq \sum_{i=0}^{2} \Omega_{i,m}(x;r).$$

Clearly, we have

$$\frac{\|\Omega_m(x;r)\|_{\mathcal{C}(X)}}{(b_n - a_n)\mathcal{P}_n} \leqq \sum_{i=0}^{2} \frac{\|\Omega_{i,m}(x;r)\|_{\mathcal{C}(X)}}{(b_n - a_n)\mathcal{P}_n}. \tag{9}$$

Now, using the above assumption about the implications in (2) to (3) and by Definition 2.2, the right-hand side of (9) is seen to tend to zero as $n \to \infty$. Consequently, we get

$$\lim_{n\to\infty} \frac{\|\Omega_m(x;r)\|_{\mathcal{C}(X)}}{(b_n - a_n)\mathcal{P}_n} = 0 \quad (\delta, r > 0).$$

Hence, the implication (1) holds true. Which completes the proof of Theorem 3.1. □

Now, by using the Definition 2.1, we present the following corollary as the consequence of Theorem 3.1.

Corollary 3.2. *Let $\mathfrak{L}_m \colon \mathcal{C}(X) \to \mathcal{C}(X)$ be a sequence of positive linear operators. Also let $f \in \mathcal{C}(X)$. Then*

$$\mathrm{stat}_{[\mathrm{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0$$

if and only if

$$\mathrm{stat}_{[\mathrm{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0,$$
$$\mathrm{stat}_{[\mathrm{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(x;x) - x\|_\infty = 0,$$

and

$$\mathrm{stat}_{[\mathrm{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(x^2;x) - x^2\|_\infty = 0.$$

We consider below an example for the sequence of random variables of positive linear operators which does not satisfy the conditions of the approximation theorems of Korovkin-type proved earlier by Jena *et al.* [20], Srivastava *et al.* [21] and Paikray *et al.* [24], but which satisfies the conditions of our Theorem 3.1. Consequently, our Theorem 3.1 is stronger than the results established earlier by both Jena *et al.* [20] and Srivastava *et al.* [21].

We now recall the operator

$$x(1+xD) \quad \left(D = \frac{d}{dx}\right),$$

that was considered by Al-Salam [33] and, very recently, by Viskov and Srivastava [34] (also see [35] the monograph by Srivastava and Manocha [36]). Here, in our Example 3.3 below, we use this operator in conjunction with the Meyer–König and Zeller operators.

Example 3.3. Let $X = [0,1]$ and we consider Meyer–König and Zeller operators $\mathfrak{M}_n(f;x)$ on $\mathcal{C}[0,1]$ given by (see [37]),

$$\mathfrak{M}_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{k+n+1}\right)\binom{n+k}{k} x^k(1-x)^{n+1}.$$

Also let $\mathfrak{L}_m\colon \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be sequence of operators defined as follows:

$$\mathfrak{L}_m(f;x) = [1+X_m]x(1+xD)\mathfrak{M}_m(f) \quad (f \in \mathcal{C}(X)), \tag{10}$$

where (X_m) is a sequence of random variables defined in Example 2.4. Now,

$$\begin{aligned}
\mathfrak{L}_m(1;x) &= [1+X_m]x(1+xD)1 = [1+X_m]x,\\
\mathfrak{L}_m(s;x) &= [1+X_m]x(1+xD)x = [1+X_m]x(1+x),
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{L}_m(s^2;x) &= [1+X_n]x(1+xD)\left\{x^2\left(\frac{n+2}{n+1}\right)+\frac{x}{n+1}\right\}\\
&= [1+X_n(x)]\left\{x^2\left[\left(\frac{n+2}{n+1}\right)x + 2\left(\frac{1}{n+1}\right)+2x\left(\frac{n+2}{n+1}\right)\right]\right\},
\end{aligned}$$

so that we have

$$\begin{aligned}
&\mathrm{stat}_{[\mathrm{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0,\\
&\mathrm{stat}_{[\mathrm{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(x;x) - x\|_\infty = 0,
\end{aligned}$$

and

$$\mathrm{stat}_{[\mathrm{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(x^2;x) - x^2\|_\infty = 0,$$

that is, the sequence $\mathfrak{L}_m(f;x)$ satisfies the conditions (2) to (3). Therefore by Theorem 3.1, we have

$$\mathrm{stat}_{[\mathrm{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f\|_\infty = 0.$$

Hence, it is deferred Cesàro and deferred weighted statistically probability convergent (or, $[D(CW)]_{sp}$ convergent). However, since (X_m) is neither Cesàro convergent nor weighted statistically convergent, so it is not deferred Cesàro and deferred weighted statistically $[D(CW)]_s$ convergent. Thus, we conclude that earlier works in [20] and [21] are not valid for the operators defined by (10), where as our Theorem 3.1 still works for the operators defined by (10).

Rate of Statistical Probability Convergence

In this section, we study the rates of the product of deferred Cesàro and deferred weighted statistical probability $[D(CW)]_{sp}$ convergence of a sequences of random variables of positive linear operator $\mathfrak{L}(f;x)$ defined on $\mathcal{C}(X)$ by using the modulus of continuity.

We begin by introducing a definition as follows.

Definition 4.1. Let (p_n) be a sequence of non-negative real numbers and let (u_n) be a positive non-increasing sequence. A given sequence (X_m) of random variables is deferred Cesàro and deferred weighted statistically probability convergent (or $[D(CW)]_{sp}$ convergent) to a random variable X with the rate $o(u_n)$ if, for every $\epsilon > 0$ and $\delta > 0$,

$$\lim_{n\to\infty} \frac{|\{m\colon m \leqq (b_n - a_n)\mathcal{R}_n \quad \text{and} \quad p_m P(|X_m - X| \geqq \epsilon) \geqq \delta\}|}{(b_n - a_n)\mathcal{R}_n u_n} = 0.$$

In this case, we may write

$$X_m - L = \mathrm{stat}_{[\mathrm{D(CW)P}]}\ o(u_n) \quad \text{or} \quad X_m - L = [D(CW)]_{sp}\ o(u_n).$$

We need a basic lemma as follows.

Lemma 4.2. *Let (u_n) and (v_n) be two positive non-increasing sequences. Let (X_m) and (Y_m) be two sequences of random variables such that*

$$X_m - X_1 = \mathrm{stat}_{[\mathrm{D(CW)P}]}\ o(u_n)$$

and

$$Y_m - X_2 = \mathrm{stat}_{[\mathrm{D(CW)P}]}\ o(v_n),$$

respectively. Then the following conditions hold true

(i) $(X_m + Y_m) - (X_1 + X_2) = \mathrm{stat}_{\mathrm{D(CW)P}}\ o(w_n)$;

(ii) $(X_m - X_1)(Y_m - X_2) = \mathrm{stat}_{\mathrm{D(CW)P}}\ o(u_n v_n)$;

(iii) $\lambda(X_m - X_1) = \mathrm{stat}_{\mathrm{D(CW)P}}\ o(u_n)$ *(for any scalar λ)*;

(iv) $\sqrt{|X_m - X_1|} = \text{stat}_{\text{D(CW)P}} \ o(u_n)$,

where

$$w_n = \max\{u_n, v_n\}.$$

Proof. In order to prove the implication (i) of Lemma 4.2, for $\epsilon > 0$ and $x \in X$, we define the following sets:

$$\mathfrak{A}_n(x;\epsilon) = |\{m \colon m \leqq (b_n - a_n)\mathcal{R}_n, \ \ p_m P(|X_m + Y_m - X_1 + X_2| \geqq \epsilon) \geqq \delta\}|,$$

$$\mathfrak{A}_{0,n}(x;\epsilon) = \left|\left\{m \colon m \leqq (b_n - a_n)\mathcal{R}_n, \ \ p_m P(|X_m - X_1| \geqq \epsilon) \geqq \frac{\delta}{2}\right\}\right|$$

and

$$\mathfrak{A}_{1,n}(x;\epsilon) = \left|\left\{m \colon m \leqq (b_n - a_n)\mathcal{R}_n, \ \ p_m P(|Y_m - X_2| \geqq \epsilon) \geqq \frac{\delta}{2}\right\}\right|.$$

Clearly, we have

$$\mathfrak{A}_n(x;\epsilon) \subseteq \mathfrak{A}_{0,n}(x;\epsilon) \cup \mathfrak{A}_{1,n}(x;\epsilon).$$

Moreover, since

$$w_n = \max\{u_n, v_n\},$$

by the condition (1) of Theorem 3.1, we obtain

$$\frac{\|\mathfrak{A}_m(x;\epsilon)\|_{\mathcal{C}(X)}}{w_n(b_n - a_n)\mathcal{R}_n} \leqq \frac{\|\mathfrak{A}_{0,n}(x;\epsilon)\|_{\mathcal{C}(X)}}{u_n(b_n - a_n)\mathcal{R}_n} + \frac{\|\mathfrak{A}_{1,n}(x;\epsilon)\|_{\mathcal{C}(X)}}{v_n(b_n - a_n)\mathcal{R}_n}.$$

Now, by the conditions (2) to (3) of Theorem 3.1, we obtain

$$\frac{\|\mathfrak{A}_n(x;\epsilon)\|_{\mathcal{C}(X)}}{w_n(b_n - a_n)\mathcal{R}_n} = 0,$$

which establishes the implication (i) of Lemma 4.2. Since the proofs of the other implications (ii) to (iv) of Lemma 4.2 are similar, we choose to omit the analogous details involved. □

We now remind that the modulus of continuity of a function $f \in \mathcal{C}(X)$ is given by

$$\omega(f,\delta) = \sup_{|y-x| \leqq \delta \colon x,y \in X} |f(y) - f(x)| \quad (\delta > 0),$$

which implies that

$$|f(y) - f(x)| \leqq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{11}$$

We state and prove a result in the form of the following theorem.

Theorem 4.3. *Let $\mathfrak{L}_m \colon \mathcal{C}(X) \to \mathcal{C}(X)$ be a sequence of random variables of positive linear operators. Assume that the following conditions hold true:*

(i) $\|\mathfrak{L}_m(1;x) - 1\|_{\mathcal{C}(X)} = \mathrm{stat}_{[\mathrm{D(CW)P}]}\ o(u_n)$,

(ii) $\omega(f, \lambda_m) = \mathrm{stat}_{[\mathrm{D(CW)P}]}\ o(v_n)$,

where

$$\lambda_m = \sqrt{\mathfrak{L}_m(\varphi^2;x)} \quad \text{and} \quad \varphi_1(y,x) = (y-x).$$

Then, for all $f \in \mathcal{C}(X)$, the following statement holds true:

$$\|\mathfrak{L}_m(f;x) - f\|_{\mathcal{C}(X)} = \mathrm{stat}_{[\mathrm{D(CW)P}]}\ o(w_n), \tag{12}$$

where $w_n = \max\{u_n, v_n\}$.

Proof. Let $f \in \mathcal{C}(X)$ and $x \in X$. Using (11), we have

$$\begin{aligned}
|\mathfrak{L}_m(f;x) - f(x)| &\leqq \mathfrak{L}_m(|f(y)-f(x)|;x) + |f(x)||\mathfrak{L}_m(1;x) - 1| \\
&\leqq \mathfrak{L}_m\left(\frac{|x-y|}{\lambda_m} + 1; x\right)\omega(f,\lambda_m) + |f(x)||\mathfrak{L}_m(1;x)-1| \\
&\leqq \mathfrak{L}_m\left(1 + \frac{1}{\lambda_m^2}\mathfrak{L}_m(x-y)^2; x\right)\omega(f,\lambda_m) + |f(x)||\mathfrak{L}_m(1;x)-1| \\
&\leqq \left(\mathfrak{L}_m(1;x) + \frac{1}{\lambda_m^2}\mathfrak{L}_m(\varphi_1^2;x)\right)\omega(f,\lambda_m) + |f(x)||\mathfrak{L}_m(1;x)-1|.
\end{aligned}$$

Putting $\lambda_m = \sqrt{L_m(\varphi^2;x)}$, we get

$$\begin{aligned}
\|L_m(f;x) - f(x)\|_{\mathcal{C}(X)} &\leqq 2\omega(f,\lambda_m) + \omega(f,\lambda_m)\|L_m(1;x)-1\|_{\mathcal{C}(X)} \\
&\qquad + \|f(x)\|\|L_m(1;x)-1\|_{\mathcal{C}(X)} \\
&\leqq \mathcal{M}\{\omega(f,\lambda_m) + \omega(f,\lambda_m)\|L_m(1;x)-1\|_{\mathcal{C}(X)} \\
&\qquad + \|L_m(1;x)-1\|_{\mathcal{C}(X)}\},
\end{aligned}$$

where

$$\mathcal{M} = \{\|f\|_{\mathcal{C}(X)}, 2\}.$$

Thus,

$$\begin{aligned}
&\left\| p_n \sum_{m=a_n+1}^{b_n} L_m(f;x) - f(x) \right\|_{\mathcal{C}(X)} \leqq \mathcal{M}\Bigg\{\omega(f,\lambda_m)p_n + \omega(f,\lambda_m) \\
&\times \left\| p_n \sum_{m=a_n+1}^{b_n} L_m(f;x) - f(x) \right\|_{\mathcal{C}(X)} + \left\| p_n \sum_{m=a_n+1}^{b_n} L_m(f;x) - f(x) \right\|_{\mathcal{C}(X)}\Bigg\}.
\end{aligned}$$

Now, under the conditions (i) and (ii) of Theorem 4.3, in conjunction with Lemma 4.2, we reach at the statement (12) of Theorem 4.3. Which completes the proof of Theorem 4.3. □

Discussion

In the last concluding section of our study, we consider various further remarks and observations associated with different results which we have established here.

Remark. Let $(X_m)_{m\in\mathbb{N}}$ be a sequence of random variables given in Example 2.4. Then, since

$$\text{stat}_{[\text{D(CW)P}]} \lim_{m\to\infty} X_m = 5 \text{ on } [0,1],$$

we have

$$\text{stat}_{[\text{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(f_i;x) - f_i(x)\|_\infty = 0 \quad (i = 0,1,2). \tag{13}$$

Thus, by applying Theorem 3.1, we can write

$$\text{stat}_{[\text{D(CW)P}]} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0, \quad (i = 0,1,2), \tag{14}$$

where

$$f_0(x) = 1, \quad f_1(x) = x, \quad \text{and} \quad f_2(x) = x^2.$$

However, since (X_m) is neither statistically convergent nor uniformly convergent in the ordinary sense, the classical and statistical Korovkin-type theorems do not work here for the operators defined by (10). Hence, clearly, this application indicates that our Theorem 3.1 is a non-trivial generalization of the classical as well as the statistical Korovkin-type theorem (see [1] and [38]).

Remark. Let $(X_m)_{m\in\mathbb{N}}$ be a sequence of random variables as given in Example 2.4. Then, since

$$\text{stat}_{[\text{D(CW)P}]} \lim_{m\to\infty} X_m = 5 \text{ on } [0,1],$$

(13) holds true. Now by applying (13) and Theorem 3.1, the condition (14) holds true. However, since the sequence (X_m) of random variables is not deferred Cesàro [20] and deferred weighted [21] statistically convergent, the results of Jena *et al.* [20] and Srivastava *et al.* (see [21]) do not work for our operator defined in (10). Thus, naturally, our Theorem 3.1 is also a non-trivial extension of the results of Jena *et al.* [20] and Srivastava *et al.* [21] (see also [24, 25]). Based upon the above results, it is concluded here that our proposed method has successfully worked for the operators defined in (10) and, therefore, it is stronger than the classical and statistical versions of the Korovkin-type approximation theorems (see [21, 24, 25]) which were established earlier.

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El producto de las convergencias estadísticas en probabilidad diferida de Cesàro con y sin peso, y sus aplicaciones a teoremas del tipo Korovkin

Resumen: En este trabajo introduciremos y estudiaremos la convergencia estadística en probabilidad de secuencias de variables aleatorias así como la convergencia estadística de secuencias de números reales, definidas sobre un espacio de Banach mediante el producto de las medias aditivas diferidas de Cesàro con y sin peso. Primero pesentaremos un teorema que conecta ambas convergencias. Basados en los métodos propuestos, a continuación probaremos un teorema de aproximación del tipo Korovkin con funciones algebraicas de prueba, para una secuencia de variables aleatorias sobre un espacio de Banach, y mostraremos que dicho teorema extiende y mejora la mayoría (sino todos) de los resultados existentes (en sus versiones clásicas y estadísticas). Más aún, presentaremos un ejemplo ilustrativo usando los operadores generalizados de Meyer-König y de Zeller para una secuencia de variables aleatorias, y así demostrar que nuestro resultado es más fuerte que sus versiones tradicionales y estadísticas. Finalmente, estimaremos la razón entre el producto de convergencias estadísticas en probabilidad diferidas de Cesàro con y sin peso, y por lo tanto estableceremos un nuevo resultado.

Palabras clave: convergencia estadística; convergencia estadística en probabilidad; media producto diferida de Cesàro con y sin peso; operador lineal y positivo; secuencia de variables aleatorias; espacio de Banach; teoremas del tipo Korovkin; razón de la convergencia estadística en probabilidad.

Produto de Cesàro diferido e convergência de probabilidade estatística ponderada diferida e suas aplicações a teoremas do tipo Korovkin

Resumo: No presente trabalho, apresentamos e estudamos a noção de convergência de probabilidade estatística para sequências de variáveis aleatórias, bem como a ideia de convergência estatística para sequências de números reais, que são definidas sobre um espaço de Banach através do produto de médias do somabilidade Cesàro diferidas e ponderadas diferidas. Primeiro estabelecemos um teorema apresentando uma conexão entre eles. Com base em nos sos métodos propostos, provamos um teorema de aproximação do tipo Korovkin com funções de teste algébrico para uma sequência de variáveis aleatórias em um espaço de Banach e demonstramos que nosso teorema efetivamente estende e melhora a maioria (se não todos) dos resultados existentes anteriormente (tanto nas versões clássicas como nas estatísticas). Além disso, um exemplo ilustrativo é apresentado aqui por meio dos operadores generalizados de Meyer-König e Zeller de uma sequência de variáveis aleatórias, a fim de demonstrar que nosso estabelecido teorema é mais forte que suas versões tradicionais e estatísticas. Por fim, estimamos a razão entre o produto das convergências de probabilidade estatística de Cesàro diferida e ponderada diferida, e em conformidade, estabelecemos um novo resultado.

Palavras-chave: convergência estatística; convergência de probabilidade estatística; média do produto Cesàro diferida e ponderada diferida; operadores lineares positivos; sequência de variáveis aleatórias; espaço de Banach; teoremas do tipo Korovkin; taxa de convergência de probabilidade estatística.

Bidu Bhusan jena

Is currently working in the Department of Mathematics, Veer Surenra Sai University of Technology, Burla Odisha, India. He has published more than 30 research papers in various National and International Journals of repute. The research area of Dr. Jena is Summability Theory, Statistical Convergence, Approximation Theory, Functional Analysis and Fourier series.

Susanta Kumar Paikray

Is currently working as an Associate Professor in the Department of Mathematics, Veer Surendra Sai University of Technology, Burla, Odisha, India. He has published more than 75 research papers in various National and International Journals of repute. The research area of Dr. Paikray is Summability theory, Statistical Convergence, Approximation Theory, Functional Analysis, Fourier series, Operations research and Inventory optimization.