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ORIGINAL ARTICLE

The Avoidance Spectrum of Alexandroff Spaces

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Abstract

In this paper we prove that every T_0 Alexandroff topological space (X, τ) is homeomorphic to the avoidance of a subspace of $(\mathrm{Spec}(\Lambda), \tau_Z)$, where $\mathrm{Spec}(\Lambda)$ denotes the prime spectrum of a semi-ring Λ induced by τ, and τ_Z is the Zariski topology. We also prove that $(\mathrm{Spec}(\Lambda), \tau_Z)$ is an Alexandroff space if and only if Λ satisfies the Gilmer property.

Keywords: Alexandroff space; avoidance spectrum; Zariski topology.

1. Introduction

Among the topological spaces that satisfy only low separation axioms (unless they are discrete) we have the so-called Alexandroff spaces (those for which arbitrary intersections of open sets are open).

If $\mathbf{2}^X$ denotes the power set of X with the product topology, in which $\mathbf{2} = \{0, 1\}$ has the discrete topology, we can view τ as a subset of $\mathbf{2}^X$. Uzcátegui and Vielma [6] proved that a topological space (X, τ) is Alexandroff if and only if τ is a closed subset of $\mathbf{2}^X$. Also, if τ is a topology on X then $\overline{\tau}$, the closure of τ in $\mathbf{2}^X$, is a topology on X, and it follows that $\overline{\tau}$ is the smallest Alexandroff topology on X containing τ.

In this paper we use a commutative semi-ring structure Λ on τ, by defining the addition and the multiplication as the union and the intersection, respectively (Section 2). For any $x \in X$ there exists a particular ideal $\Phi_\tau(x)$ of τ, namely $\{U \in \tau : x \notin U\}$, which we call "the avoidance" ideal of x. It turns out that, as we show in Section 3, $\Phi_\tau(x)$ is a prime ideal of τ and it plays an important role in comparing Alexandroff topologies that satisfy the axiom T_0 with a particular subspace of the prime spectrum of τ with the Zariski topology. More specifically, we find a homeomorphism between X and the collection of all avoidance ideals of elements of X. Finally, in Section 4, we give a solution in the context of semi-rings, to a problem proposed by Gilmer [1], that is a characterization of the rings that satisfy the property that for every family of prime ideals $\{\mathfrak{p}_\alpha : \alpha \in J\}$ such that $\cap_{\alpha\in J} \mathfrak{p}_\alpha \subset \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, then there exists an $\alpha \in J$ such that $\mathfrak{p}_\alpha \subset \mathfrak{p}$ (Theorem 4.2).

2. Preliminaries

If (X, τ) is a topological space, we say that τ is an *Alexandroff topology* if any arbitrary intersection of open sets is open, see Munkres [5].

One way to approach the study of Alexandroff spaces is by considering an algebraic structure on τ. In fact, for any topological space (X, τ), we can consider τ as a commutative semi-ring Λ, where addition and multiplication are given by the union and the intersection, respectively, and the corresponding neutral elements being $\emptyset$ and X, see Golan [2].

In the case of Alexandroff topologies we get some interesting results by working with some particular ideals of τ. For any $x \in X$, we consider the set

$$\Phi_\tau(x) = \{U \in \tau : x \notin U\}$$

which is, obviously, an ideal of τ. We call this $\Phi_\tau(x)$ the *avoidance ideal* of x (because, somehow, its members avoid x.) Our first characterization of Alexandorff spaces is obtained by considering the *avoidance spectrum* (or simply, *avoidance*) of X, which is given by

$$\mathcal{A}_\tau(X) = \{\Phi_\tau(x) : x \in X\}.$$

The proof of such a result involves the *principal ideal generated* by any set $A \in \tau$, that is

$$\langle A \rangle = \{U \in \tau : U \subset A\}.$$

In this setting we obtain the following property of Alexandroff spaces.

Lemma 2.1. *If τ is an Alexandroff topology on X, then every family of principal ideals is closed under arbitrary intersections.*

Proof. Suppose that (X, τ) is Alexandroff and $\{\mathfrak{i}_\alpha : \alpha \in J\}$ is a family of principal ideals of τ. Therefore, for each $\alpha \in J$ we have $\mathfrak{i}_\alpha = \langle U_\alpha \rangle$ for some $U_\alpha \in \tau$. Then

$$\bigcap_{\alpha \in J} \mathfrak{i}_\alpha = \bigcap_{\alpha \in J} \langle U_\alpha \rangle = \left\langle \bigcap_{\alpha \in J} U_\alpha \right\rangle. \qquad \square$$

Some of the main properties of avoidance ideals are given in the following propositions.

Theorem 2.2. The avoidance ideal $\Phi_\tau(x)$ is a prime ideal, for all $x \in X$.

Proof. Clearly, $\Phi_\tau(x)$ is an ideal of τ. If $U, V \in \tau$ and $U \cap V \in \Phi_\tau(x)$, then $x \notin U \cap V$. Therefore, either $x \notin U$ or $x \notin V$, which implies that either $U \in \Phi_\tau(x)$ or $V \in \Phi_\tau(x)$, thus $\Phi_\tau(x)$ is a prime ideal of τ. $\square$

The proof of the next lemma is almost obvious, especially if we note that if $x, y \in X$, then $\Phi_\tau(x) \subset \Phi_\tau(y)$ if and only if $\overline{\{y\}} \subset \overline{\{x\}}$.

Lemma 2.3. *Let τ be a topology on X. Then the following conditions are equivalent:*

1. *The topology τ is* T_0.
2. *If $x \neq y$, then $\Phi_\tau(x) \neq \Phi_\tau(y)$.*

Now, we present a result involving spaces that satisfy a special property related to intersections of ideals.

Theorem 2.4. Let τ be an Alexandroff topology on X and x an arbitrary point. For any family $\{\mathfrak{i}_\alpha : \alpha \in J\}$ of ideals of τ, if

$$\bigcap_{\alpha \in J} \mathfrak{i}_\alpha \subset \Phi_\tau(x),$$

then, there exists an $\alpha \in J$ such that $\mathfrak{i}_\alpha \subset \Phi_\tau(x)$.

Proof. Let $x \in X$ and $\{\mathfrak{i}_\alpha : \alpha \in J\}$ a family of ideals of τ such that

$$\bigcap_{\alpha \in J} \mathfrak{i}_\alpha \subset \Phi_\tau(x).$$

If $\mathfrak{i}_\alpha \not\subseteq \Phi_\tau(x)$ for all $\alpha \in J$, then for each $\alpha \in J$, there is a $V_\alpha \in \mathfrak{i}_a$ such that $V_\alpha \notin \Phi_\tau(x)$. Therefore $x \in V_\alpha$ for all $\alpha \in J$. If $V = \cap_{\alpha \in J} V_\alpha$ we have that $V \in \tau$ and $x \in V$.

On the other hand $V \in \mathfrak{i}_\alpha$ for all α and $V \in \tau$, but this implies that $x \notin V$, which is a contradiction. □

3. Alexandroff spaces and the avoidance spectrum

The main theorem of this paper shows a comparison between a T_0 Alexandroff topological space (X, τ) and $\mathcal{A}_\tau(X)$, the latest seen as a subspace of the prime spectrum of τ with Zariski topology. Recall that for a semi-ring Λ the *prime spectrum*, or simply, the *spectrum* of Λ is the collection

$$\mathrm{Spec}(\Lambda) = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } \Lambda\}.$$

If $\mathfrak{i}$ is an ideal of Λ we denote by $\nu(\mathfrak{i})$ the following set

$$\{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } \Lambda \text{ and } \mathfrak{i} \subset \mathfrak{p}\}.$$

The *Zariski topology* τ_Z is the one on $\mathrm{Spec}(\Lambda)$ whose closed sets are of the form $\nu(\mathfrak{i})$. For more details see Hungerford [3] and Lang [4].

Let us note that when this construction is carried out for a topological space (X, τ), then for every $x \in X$, we have that $\Phi_\tau(x) \in \mathrm{Spec}(\tau)$. Finally, in this context, we obtain that if τ is a T_0 Alexandroff topology, then the map from X into $\mathcal{A}_\tau(X)$, $x \mapsto \Phi_\tau(x)$ is a homeomorphism, as will be proven later. The next example illustrates this fact.

Example 3.1. The collection

$$\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$$

is a T_0 Alexandroff topology on $X = \{0, 1, 2\}$. Then $\mathrm{Spec}(\tau) = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, where

$$\begin{aligned}
\mathfrak{a} &= \{\emptyset, \{1\}\}, \\
\mathfrak{b} &= \{\emptyset, \{2\}\}, \\
\mathfrak{c} &= \{\emptyset, \{1\}\}, \{2\}\}, \{1, 2\}\}.
\end{aligned}$$

Then $\mathrm{Spec}(\tau) = \{\mathfrak{o}, \mathfrak{x}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \tau\}$, and

$$\begin{aligned}
\eta(\mathfrak{o}) &= \{\mathfrak{o}, \mathfrak{x}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \tau\}, \\
\eta(\mathfrak{x}) &= \{\mathfrak{x}\}, \\
\eta(\mathfrak{a}) &= \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}, \\
\eta(\mathfrak{b}) &= \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}, \\
\eta(\mathfrak{c}) &= \{\mathfrak{c}, \mathfrak{d}\}, \\
\eta(\mathfrak{d}) &= \{\mathfrak{d}\}.
\end{aligned}$$

On the other hand, we have that the Zariski topology is

$$\tau_Z = \{\emptyset, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{b}\}, \mathcal{A}_\tau(X)\}.$$

Therefore, we have

$$\Phi_\tau(0) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} = \mathfrak{c},$$
$$\Phi_\tau(1) = \{\emptyset, \{2\}\} = \mathfrak{b},$$
$$\Phi_\tau(2) = \{\emptyset, \{1\}\} = \mathfrak{a}.$$

It is clear that the map $\psi : X \to \mathcal{A}_\tau(X)$, defined by $\psi(x) = \Phi_\tau(x)$, is an open bijection. The next theorem will show us that ψ is a homeomorphism.

Theorem 3.2. A T_0 Alexandroff topological space (X, τ) is homeomorphic to $\mathcal{A}_\tau(X)$ with the subspace topology from the Zariski space $(\mathrm{Spec}(\tau), \tau_Z)$.

Proof. We prove that the function $\psi : X \to \mathcal{A}_\tau(X)$ defined by $\psi(x) = \Phi_\tau(x)$ is a homeomorphism. Since τ is a T_0 topology, we have that ψ is injective, and therefore it is bijective. If $\mathfrak{i}$ is an ideal of τ, then $\nu(\mathfrak{i}) \cap \mathcal{A}_\tau(X)$ is a closed set in $\mathcal{A}_\tau(X)$.

Then, we have

$$\begin{aligned}\psi^{-1}(\nu(\mathfrak{i}) \cap \mathcal{A}_\tau(X)) &= \{x \in X : \mathfrak{i} \subset \Phi_\tau(x)\}\\ &= \{x \in X : x \notin U \text{ for all } U \in \mathfrak{i}\}\\ &= \left\{x \in X : x \notin \bigcup_{U \in \mathfrak{i}} U\right\}\\ &= X \setminus \bigcup_{U \in \mathfrak{i}} U,\end{aligned}$$

which is closed in X and therefore, ψ is continuous.

Now let $U \in \tau$, we prove that $\psi(U)$ is open in $\mathcal{A}_\tau(X)$. Since $U \notin \psi(x)$ for every $x \in U$ and $U \in \psi(y)$ for all $y \notin U$, this means that $\langle U \rangle \subset \Phi_\tau(y)$ for all $y \notin U$. Then we have that $\psi(z) \in \nu(\langle U \rangle)$ for all $z \notin U$, and $\psi(z) \notin \nu(\langle U \rangle)$ for all $z \in U$. Therefore

$$\mathcal{A}_\tau(X) \setminus \psi(U) = \{\psi(z) : z \notin U\} = \nu(\langle U \rangle).$$

Thus $\psi(U)$ is open and ψ is a homeomorphism. □

Let us note that if the space (X, τ) is T_0, $(X, \overline{\tau})$ is T_0. Then we have the following result.

Corollary 3.3. If τ is a T_0 topology on X, then the space $(X, \overline{\tau})$ is homeomorphic to $\mathcal{A}_{\overline{\tau}}(X)$.

Finally, the next example illustrates the fact that the hypothesis about the topology τ being T_0 in Theorem 3.2 cannot be dropped.

Example 3.4. If $X = \{0, 1, 2\}$, then the collection

$$\tau = \{\emptyset, \{2\}, X\}$$

is an Alexandroff topology on X which does not satisfy the axiom T_0, and we have

$$\Phi_\tau(0) = \{\emptyset, \{2\}\},$$
$$\Phi_\tau(1) = \{\emptyset, \{2\}\},$$
$$\Phi_\tau(2) = \{\emptyset\}.$$

Then X and $\mathcal{A}_\tau(X) = \{\{\emptyset\}, \{\emptyset, \{2\}\}\}$ are not homeomorphic.

4. Alexandroff spaces and the Gilmer property

In this section we investigate some additional aspects of $(\mathrm{Spec}(\Lambda), \tau_Z)$, when τ_Z is an Alexandroff topology. The following results are related to a problem set by Gilmer [1]. The problem is about obtaining characterizations for commutative rings Λ such that:

(G) for every family of prime ideals $\{\mathfrak{p}_\alpha : \alpha \in J\}$ such that $\cap_{\alpha\in J}\mathfrak{p}_\alpha \subset \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, then there exists an $\alpha \in J$ such that $\mathfrak{p}_\alpha \subset \mathfrak{p}$.

In the context of our investigations we work with semi-rings and call (G) "the Gilmer property." Let us start by setting some notation. Denote by $\eta(\mathfrak{i})$ the nil radical of $\mathfrak{i}$, that is, the intersection of all prime ideals containing $\mathfrak{i}$. It is clear that $\nu(\mathfrak{i}) = \nu(\eta(\mathfrak{i}))$. When $\eta(\mathfrak{i}) = \mathfrak{i}$ we say that $\mathfrak{i}$ is *semi-prime*. Let us note that for any ideal $\mathfrak{i}$ there exists a unique semi-prime ideal containing $\mathfrak{i}$, namely $\eta(\mathfrak{i})$. Therefore, if τ_Z is the Zariski topology on $\mathrm{Spec}(\Lambda)$, then the τ_Z-closed sets are of the form $\nu(\mathfrak{i})$ for some (unique) semi-prime ideal $\mathfrak{i}$ of Λ and the τ_Z-open sets in $\mathrm{Spec}(\Lambda)$ are of the form

$$D_0(\mathfrak{i}) = \mathrm{Spec}(\Lambda) \setminus \nu(\mathfrak{i}),$$

and uniqueness of $\mathfrak{i}$ for $\eta(\mathfrak{i})$ implies it for $D_0(\mathfrak{i})$. The following proposition establishes a relation between the families

$$\{\nu(\mathfrak{i}_\alpha)\}_{\alpha\in J} \quad \text{and} \quad \{\eta(\mathfrak{i}_\alpha)\}_{\alpha\in J},$$

for a set of indexes J.

Lemma 4.1. *Let Λ be a semi-ring and let $\{\mathfrak{i}_\alpha\}_{\alpha\in J}$ be a family of ideals in Λ, such that*

$$\bigcup_{\alpha\in J} \nu(\mathfrak{i}_\alpha) = \nu(\mathfrak{j})$$

for some ideal $\mathfrak{j}$ of Λ. Then

$$\eta(\mathfrak{j}) = \bigcap_{\alpha\in J} \eta(\mathfrak{i}_\alpha).$$

Proof. Since $\nu(\mathfrak{i}_\alpha) \subset \nu(\mathfrak{j})$ for all $\alpha \in J$, then $\eta(\mathfrak{j}) \subset \eta(\mathfrak{i}_\alpha)$, therefore

$$\eta(\mathfrak{j}) \subset \bigcap_{\alpha\in J} \eta(\mathfrak{i}_\alpha).$$

On the other hand, suppose that $x \in \bigcap_{\alpha\in J} \eta(\mathfrak{i}_\alpha)$ and $x \notin \eta(\mathfrak{j})$. Then, there exists a prime ideal $\mathfrak{p}$ of Λ, such that $\mathfrak{p} \in \nu(\mathfrak{j})$ and $x \notin \mathfrak{p}$. Now, by hypothesis, there exists an $\alpha \in J$ such that $\mathfrak{p} \in \nu(\mathfrak{i}_\alpha)$ and, since $x \in \eta(\mathfrak{i}_\alpha) \subset \mathfrak{p}$, we get a contradiction. □

Theorem 4.2. Let Λ be a commutative semi-ring. Then $(\mathrm{Spec}(\Lambda), \tau_Z)$ is an Alexandroff space if and only if Λ satisfies the Gilmer property.

Proof. Suppose that $(\mathrm{Spec}(\Lambda), \tau_Z)$ is an Alexandroff space and $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$, such that

$$\bigcap_{\alpha} \mathfrak{p}_\alpha \subset \mathfrak{p},$$

for a family $\{\mathfrak{p}_\alpha : \alpha \in J\}$ of elements of $\mathrm{Spec}(\Lambda)$. Now, since $\nu(\mathfrak{p}_\alpha)$ is closed in $\mathrm{Spec}(\Lambda)$, then we have that $\cup_{\alpha\in J}\nu(\mathfrak{p}_\alpha)$ is closed. Therefore, there exists a prime ideal $\mathfrak{j}$ of Λ, such that

$$\bigcup_{\alpha\in J} \nu(\mathfrak{p}_\alpha) = \nu(\mathfrak{j}).$$

Then, Lemma 4.1 implies that

$$\eta(\mathfrak{j}) = \bigcap_{\alpha\in J} \eta(\mathfrak{p}_\alpha) = \bigcap_{\alpha\in J} \mathfrak{p}_\alpha.$$

Then $\mathfrak{j} \subset \mathfrak{p}$ and therefore

$$\mathfrak{p} \in \bigcup_{\alpha\in J} \nu(\mathfrak{p}_\alpha),$$

which implies that $\mathfrak{p} \in \nu(\mathfrak{p}_\alpha)$ for some $\alpha \in J$. Then $\mathfrak{p}_\alpha \subset \mathfrak{p}$ and Λ satisfies the Gilmer property.

Conversely, suppose that Λ satisfies the Gilmer property and let $\{U_\alpha : \alpha \in J\}$ be a subset of τ_Z. We prove that if

$$U = \bigcap_{\alpha\in J} U_\alpha,$$

then $U \in \tau_Z$. In fact, for every $\alpha \in J$ there exists a semi-prime ideal $\mathfrak{i}_\alpha$ such that $U_\alpha = D_0(\mathfrak{i}_\alpha)$. Now, let us prove that

$$\nu\left(\bigcap_{\alpha\in J} \mathfrak{i}_\alpha\right) = \bigcup_{\alpha\in J} \nu(\mathfrak{i}_\alpha).$$

It is clear that

$$\bigcup_{\alpha\in J} \nu(\mathfrak{i}_\alpha) \subset \nu\left(\bigcap_{\alpha\in J} \mathfrak{i}_\alpha\right),$$

since $\bigcap_{\alpha\in J} \mathfrak{i}_\alpha \subset \mathfrak{i}_\alpha$ for all $\alpha \in J$. On the other hand, take $\mathfrak{p} \in \nu(\bigcap_{\alpha\in J} \mathfrak{i}_\alpha)$, then

$$\bigcap_{\alpha\in J} \mathfrak{i}_\alpha \subset \mathfrak{p}.$$

Now, since each $\mathfrak{i}_\alpha$ is semi-prime, then every $\mathfrak{i}_\alpha$ is an intersection of prime ideals and, therefore, $\mathfrak{p}$ contains an intersection of prime ideals. By hypothesis, there is a prime ideal $\mathfrak{q}$ such that $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \in \nu(\mathfrak{i}_\alpha)$ for some $\alpha \in J$. Then

$$\mathfrak{p} \in \bigcup_{\alpha\in J} \nu(\mathfrak{i}_\alpha)$$

and

$$U = \bigcap_{\alpha\in J} D_o(\mathfrak{i}_\alpha) = D_0\left(\bigcap_{\alpha\in J} \mathfrak{i}_\alpha\right),$$

which implies that U is τ_Z-open and therefore $(\mathrm{Spec}(\Lambda), \tau_Z)$ is an Alexandroff space. □

5. Conflict of Interest

The authors certify that they have no affiliations with, or involvement in, any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent arrangements), or non (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript. The authors declare that there are no conflicts of interest.

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El espectro de evasión de espacios de Alexandroff

Resumen: En este artículo demostramos que todo espacio topológico T_0 de Alexandroff (X, τ) es homeomorfo a la evasión un subespacio de $(\mathrm{Spec}(\Lambda), \tau_Z)$, donde $\mathrm{Spec}(\Lambda)$ denota el espectro primo de un semianillo Λ inducido por τ, y τ_Z es la topología de Zariski. También demostramos que $(\mathrm{Spec}(\Lambda), \tau_Z)$ es un espacio de Alexandroff si y solo si Λ satisface la propiedad de Gilmer.

Palabras Clave: espacio de Alexandroff; espectro de evasión; topología de Zariski.

O espectro de evitação dos espaços de Alexandroff

Resumo: Neste artigo provamos que todo espaço topológico T_0 de Alexandroff (X, τ) é homeomórfico à evitação de um subespaço de (Spec(Λ), τ_Z), onde Spec(Λ) denota o espectro principal de um semianel Λ induzido por τ, e τ_Z é a topologia de Zariski. Também provamos que (Spec(Λ), τ_Z) é um espaço de Alexandroff se e somente se Λ satisfaz a propriedade de Gilmer.

Palavras-chave: espaço de Alexandroff; espectro de evitação; topologia de Zariski.

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