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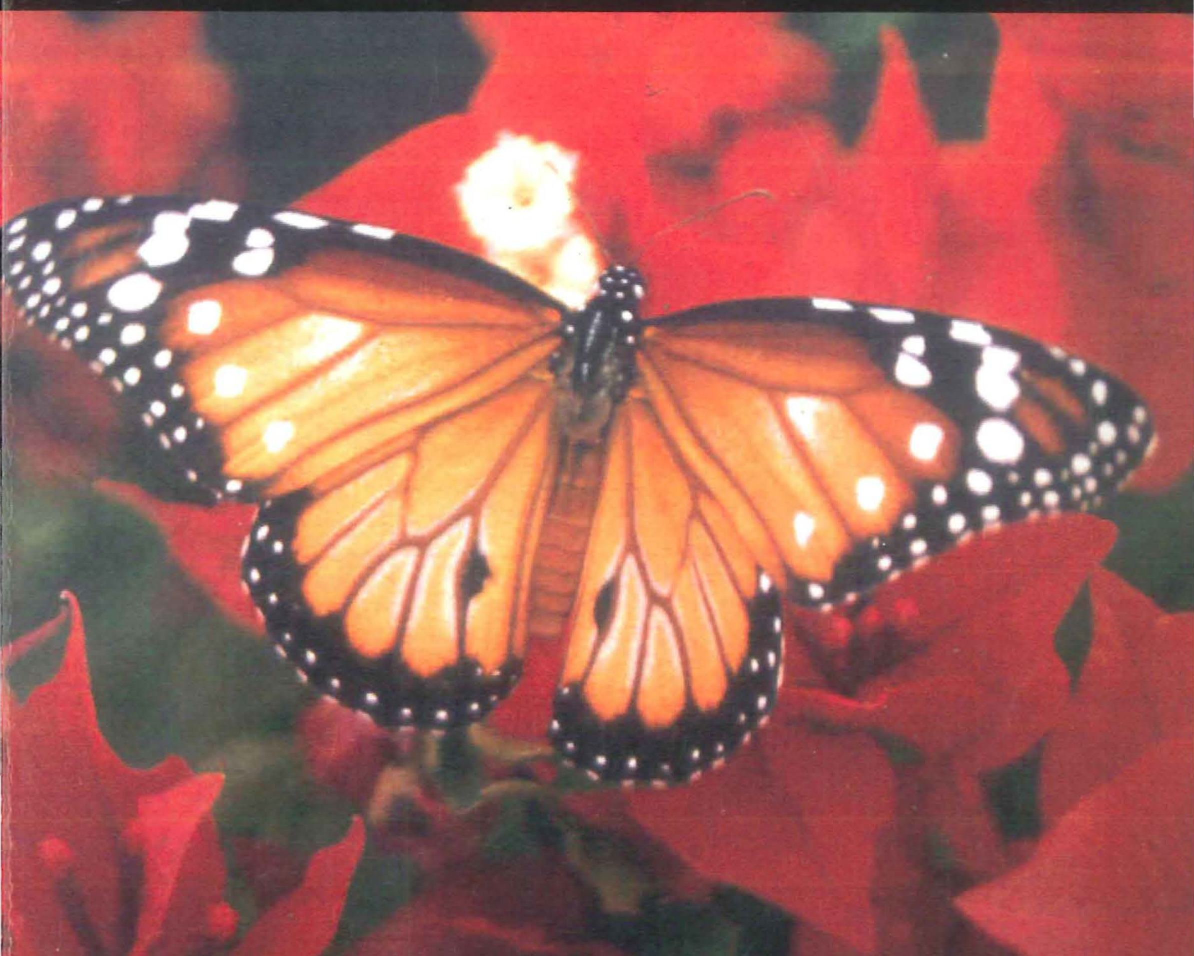

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NEW MATHEMATICAL FINDINGS BY SECONDARY STUDENTS

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ABSTRACT

Dynamic geometry software is changing the way we teach and learn geometry. In this article, we review a selection of new results, published during the last five years, which were found by secondary students using mostly this type of software. Some of these solutions represent new mathematical findings. The amount and depth of the problems solved in such a short time gives an idea of the potential for growth that technology, and in particular this software brings to the mathematics classroom.

RESUMEN

El *software* de geometría dinámica está cambiando el proceso de enseñanza y aprendizaje en geometría. En este artículo se presenta una selección de problemas resueltos por estudiantes de secundaria, durante los últimos cinco años, usando principalmente este tipo de *software*. Algunas de estas soluciones representan resultados nuevos. La cantidad y el alcance de los problemas resueltos por estudiantes tan jóvenes, en un período tan corto de tiempo, dan una idea del potencial que la tecnología en general y este tipo de *software* en particular aportan a la enseñanza de las matemáticas.

The introduction during this decade of dynamic geometry software such as The Geometer Sketchpad and Cabri has made possible to drastically change the way we teach and learn geometry. The students using this software not only may develop geometric intuition by constructing elementary figures, but they can also test or uncover their defining properties. Moreover, the students can deform or transform their constructions and observe which properties remain invariant. This capabilities allows them to explore, discover, test and conjecture new properties and relationships, anticipating the need for formal proofs.

We remark that the impact of dynamic geometry is extending beyond the teaching and learning of geometry. Researchers [Vonder Embse et all, 1998] are beginning to use dynamic geometry to facilitate the understanding of fundamental concepts of algebra and calculus.

Although the introduction of dynamic geometry software in the geometry classroom has just begun and has not reached many schools, we can already see some of its effects. Thus, during the last five years there has been an increase in the number of mathematical findings by secondary students. I have used this fact in some presentations to illustrate how technology is empowering students to do mathematics like never before, that is, to explore graphically or numerically and test ideas, to find mathematical relationships, and to conjecture results. Invariably, whenever I have mentioned any of these findings, no matter the level of the presentation or the country, the audience (consisting typically of math educators) has reacted positively, and I have had numerous petitions about the details of the students' findings. So I put together this presentation for two reasons: I) to inform math educators and provide them with concrete evidence of what the students can do with the right tools if they are properly challenged, and II) to try to gather additional information on some other students' findings that I may not be aware of.

Next, I review a selection of problems solved by secondary students. Some of these solutions represent new mathematical findings. Our 1st example presents a new result obtained by combining the use of the Geometer Sketchpad and a TI-82 graphing calculator. The persistence and insight displayed by the author shows an unusual mathematical maturity for a 9th grader.

Example 1: On an extension of Marion Walter's Theorem

In the February issue of 1992, the cover of the *Mathematics Teacher*, created by William Johnston, showed that: *If selected trisection points of an equilateral triangle are joined to the opposite vertices, the resulting central triangle had area 1/7 the area of the original triangle* (see figure 1.a). This prompted the publication of Marion Walter's theorem in the Mathematics Teacher [Cuoco, Goldenberg, and Mark, 1993], that is, *If the trisection points of the sides of any triangle are connected to the opposite vertices, the resulting hexagon has area one-tenth the area of the original triangle* (as illustrated in figure 1.b).

In the fall of 1993, Frank D. Nowosielski a teacher in Patapsco High School, Baltimore County, Maryland, proposed to his ninth-grade geometry class to "rediscover" Marion Walter's theorem using the Geometer Sketchpad. One of the students in this class, Ryan Morgan, after verifying the theorem, became interested in finding out what would it happen if the sides of the triangle were partitioned into more than three congruent segments. He and his teacher later termed this process ***n-secting*** [Watanabe et all, 1996].

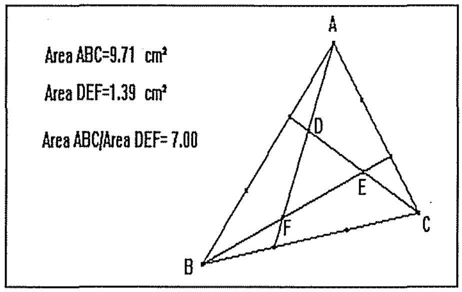

FIGURE 1.a

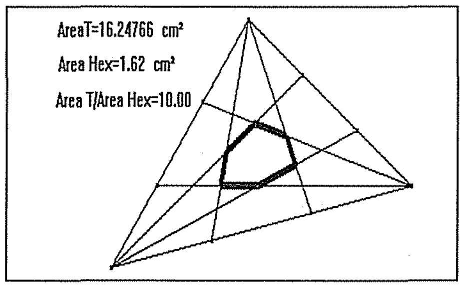

FIGURE 1.b

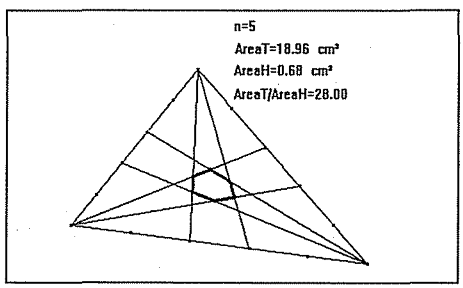

FIGURE 1.c

First, Ryan noticed that in order to have "two" central points, the side had to be subdivided in an odd number of parts. Then, using the Geometer Sketchpad, Ryan found that the ratio of the area of the triangle to the central hexagon was also constant when n was an odd number greater than 3.

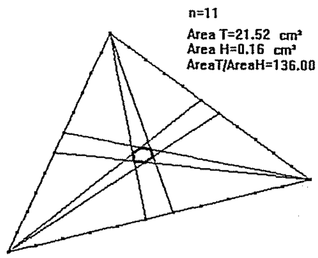

FIGURE 2.a

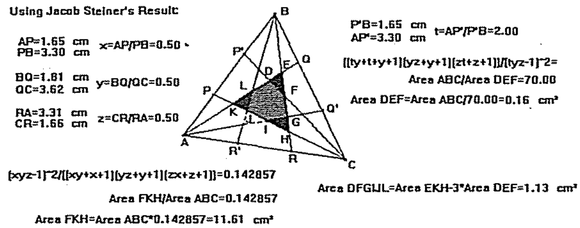

FIGURE 2.b

Figures 1.c, 2.a show the value of this constant for $n = 5{,}11$. The invariance of these ratios can be easily observed by grabbing one of the vertices of any of these triangles and deforming it. Table 1 contains the ratios determined for the first six values of n.

n-sections	3	5	7	9	11	13
Ratio of areas	10:1	28:1	55:1	91:1	136:1	190:1

Table 1

After plotting these points using a graphing calculator, Ryan noticed the linear tendency and decided to fit the data using a regression line, the result allowed him to formulate his result. Ryan's conjecture:

"For n odd, if the central n-section points of the sides of any triangle are connected to the opposite vertices, the ratio of the area of the original triangle to the area of the resulting hexagon is $(9n^2-1)/8$ to 1.

After his discovery, Ryan presented his conjecture at Towson State University. As the faculty notice, the proof of his finding can be easily obtained (see figure 2.b) using the following theorem due to Jacob Steiner (1796-1863).

If the sides AB, BC, and CA of triangle ABC are divided at P, Q and R in the respective ratios x:1, y:1, and z:1, the ratio of the area of the triangle formed by intersecting the segments AP, BQ, and CR to the area of triangle ABC is

$$K=(xyz-1)^2/[(xy+x+1)(yz+y+1)(zx+z+1)]$$

Ryan rejected some hints on how to prove his result and later submitted a proof. To this day there is no reference found in the literature to a result similar to Ryan's. The teacher, Frank D. Nowosielski, have since retired. He has fond memories of that honor geometry group. Ryan is now a senior at University of Maryland majoring in computer science

It is clear that this result would have been very difficult without a tool that allows to draw polygonal figures and obtain areas quickly and accurately, and to dynamically deform the figures to observe invariant properties.

Example 2

Arne Engerbretsen, a well known teacher at Greendale H.S., Greendale, WI, proposed to his freshmen geometry class the following problem: Given a triangle ABC, find a point P such that the sum of the distances from the three vertices, PA+PB+PC, is minimized. The recount that follows can be found in [Vonder Embse, C., Engerbretsen, A., 1996]

Two of the students Bridget & Connie, proceeded the first day, by using trial and error, to place a point P inside of the triangle ABC and measure the distances PA, PB, and PC.

Then they calculated the sum S=PA+PB+PC and move P around to find a location for P that will minimize S. Then the second day they look for a systematic way for determining the point. However, five minutes before the end of the class, they had not found one. They had constructed over the sides of the given triangle, equilateral triangles with their centroids. Partly out of desperation, Connie suggested reflecting vertex B over the line joining the closest centroids of the equilateral triangles. The point seems to coincide with the one obtained by trial & error! The third day, they eagerly continued their investigation confirming that their method worked for all three vertices of every acute triangle they had time to explore. When the time to present their findings came, they reported their discovery with pride and confidence.

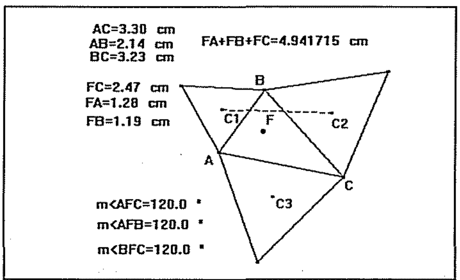

FIGURE 3.a

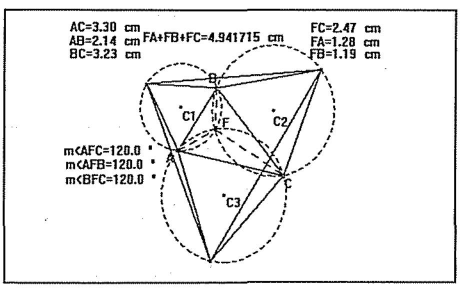

FIGURE 3.b

Arne & Chuck, very appropriately, quoted a reference [Polya, 1954] that points to math exploration, investigation, and conjecturing as the critical 1st steps in the learning process that concludes with the rigor of the proof.

"You have to guess a mathematical theorem before you prove it; you have to guess the idea of the proof before you carry through the details. You have to combine observations and follow analogies; you have to try and try again. The result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing. If the learning of math reflects at any degree the invention of math, it must have a place for guessing, for plausible inference".

Example 3

(The GlaD Construction) In June, 1995, Charles H. Dietrich teacher at Greens Farms Academy, pose to his 9th grade students Dave Goldenheim & Dan Litchfield the well known problem of how to subdivide a given segment in n equal segments. However he added the condition of doing it without using a compass [Dietrich et all, 1997].

In a few hours the student produced two constructions, one for n odd and another for n even, that can be put together as illustrated in figure 3.a [Wilson, 1998]. They also found the subdivision of figure 3.b, which with the help of their teacher recognized as having ratios that follow the Fibonacci sequence.

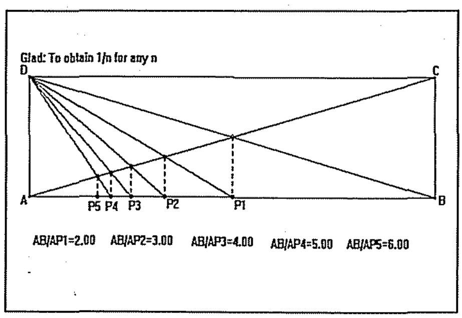

FIGURE 4.a

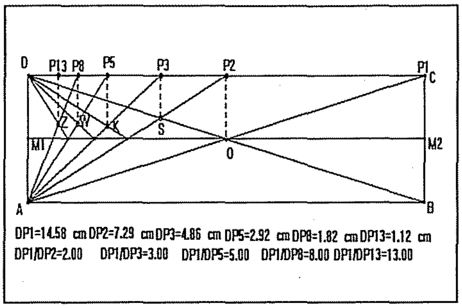

FIGURE 4.b

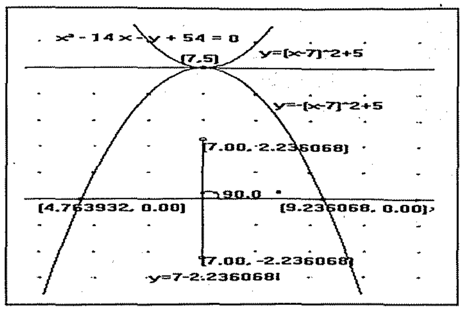

FIGURE 5.a

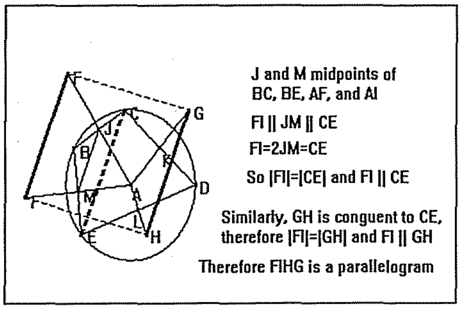

FIGURE 5.b

The solution to this problem does not appear in traditional textbooks, however it has been previously published [Leslie, 1811]. The construction that these students found is nevertheless an outstanding accomplishment.

It should be noted that Mr. Dietrich did not believe that these students would solve the problem, yet the students prove him wrong only because he dare to challenge them!

Example 4

We found in the College Mathematical journal a graphical representation of the imaginary solutions of a second-degree equation as depicted in figure 4.a. The graph was attributed to Shaun Piper a student at St. Paul's School, in Concord, New Hampshire. It shows that by reflecting the graph of we obtain the graph of whose x-intercepts are. By rotating 90° the segment defined by these intercepts, we obtain a segment, whose end points are.

Example 5

Figure 4.b shows the solution of the fall of 1999 Sketchpad puzzler winner by Lori Sommars, a 10th grader at Wheaton North High School, Wheaton, Illinois. The problem posed by Michael de Villiers was to find the type of quadrilateral formed by joining the points obtained by reflecting the circuncenter of a cyclic quadrilateral over each of its four sides. In addition to Lori three other students solved the problem.

The limitation on the length of this article precludes us from including two other relevant new results (Key Curriculum Press, Key Innovator Award Winners, 1996-97, 1999-2000) for which we have not yet seen the proof. The 1st one is the *trisoid* or locus of all points in the plane whose distance from three points is a fixed sum, found by Bilge Dermiköz a junior at Robert College a leading high school in Istanbul. Anthony Varilly at British school of Costa Rica found the 2nd result, that is a family of curves defined by the loci of the incenters of triangles with a fixed Euler line. These curves fill the Guinard circle [Guinard, 1984] without intersecting each other.

CONCLUSION

This selection of problems solved by secondary students in just a few years gives an idea of the potential for growth that technology in general, and in particular dynamic geometry software brings to the classroom. This trend of new findings by students will continue as long as we teachers dare to challenge them.

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